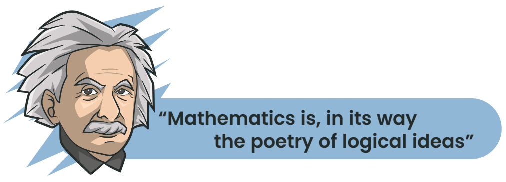


RELATIONS

(1) Types of Relations

1. Empty Relation
A relation in which no element of A is related to any other element of A, i.e., R=φ ⊂ A×A.

2. Universal Relation
A relation in which each element of A is related to every element of A, i.e., R=A×A.

3. Identity Relation
A relation in which each element is related to itself only. $I = \{(a,a), a \in A\}$

4. Reflexive Relation:
$(a, a) \in R$, for every $a \in A$.

5. Symmetric Relation:
$(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

6. Transitive Relation:
$(a_1, a_2) \in R$ & $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

7. Equivalence Relation :
A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric & transitive.

8. Inverse Relation
Inverse relation of R from A to B, denoted by R^{-1}, is a relation from B to A is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

9. Asymmetric Relation
$(x,y) \in R \Rightarrow (y,x) \notin R$

10. Antisymmetric: A relation is antisymmetric if:
- For all $x, y \in X[(x,y) \in R \;\&\; (y,x) \in R] \Rightarrow x = y$
- For all $x, y \in X[(x,y) \in R \;\&\; x \neq y] \Rightarrow (y,x) \notin R$

11. Irreflexive
R is irreflexive iff
$\forall a \in A, ((a,a) \notin R)$

12. Partial order relation
R is a partial order, if R is Reflexive, Antisymmetric and Transitive.

2. EXAMPLE:

A = {1, 2, 3, 4}. Identify the properties of relations.

$R_1 = \{(1,1),(2,2),(3,3),(2,1),(4,3),(4,1),(3,2)\}$

$R_2 = A \times A,\ R_3 = \phi,\ R_4 = \{(1,1),(2,2),(3,3),(4,4)\}$

$R_5 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(4,3),(3,4)\}$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	✗	✗	✗	✓	✗	✗
R_2	✓	✓	✗	✗	✗	✓
R_3	✗	✓	✓	✓	✓	✓
R_4	✓	✓	✗	✓	✗	✓
R_5	✓	✓	✗	✗	✗	✓

NOTE
If A={1,2}, a relation R={(1,2)} on A is a transitive relation.
using the similar argument a relation R = {(x, y) : x is wife of y} is transitive, where as R = {(x, y) : x is father of y} is not transitive.

3. PROPERTIES

1. R is not reflexive does not imply R is irreflexive. Counter example: $A = \{1,2,3\}, R = \{(1,1)\}$

2. R is asymmetric implies that R is irreflexive. By definition, for all $a, b \in A, (a,b) \in R$ and $(b,a) \notin R$ This implies that for all $(a,b) \in R, a \neq b$ Thus, for all $a \in A, (a,a) \notin R$ Therefore, R is irreflexive.

3. R is not symmetric does not imply R is antisymmetric. Counter example: $A = \{1,2,3\}, R = \{(1,2),(2,3),(3,2)\}$

4. R is not symmetric does not imply R is asymmetric. Counter example: $A = \{1,2,3\}, R = \{(1,2),(2,2)\}$

5. R is not antisymmetric does not imply R is symmetric. Counter example: $A = \{1,2,3\}, R = \{(1,2),(2,3),(3,2)\}$

6. R is reflexive implies that R is not asymmetric. By definition, for all $a \in A, (a,a) \in R$ This implies that, both (a, b) and (b, a) are in R when a = b. Thus, R is not asymmetric.

(4) COUNTING OF RELATION

- Number of relations from set A to B = 2^{mn}, where $|A| = m, |B| = n$
- Number of Identity relation on a set with 'n' elements = 1
- Number of reflexive relation set on a set with 'n' elements = $2^{n(n-1)}$
- Number of Symmetric relation set on a set with 'n' elements = $2^{n(n+1)/2}$
- The number of antisymmetric binary relations possible on A is $2^n \cdot 3^{(n^2-n)/2}$
- The number of binary relation on A which are both symmetric and antisymmetric is 2^n.
- The number of binary relation on A which are both symmetric and asymmetric is 1.
- The number of binary relation which are both reflexive and antisymmetric on the set A is $3^{(n^2-n)/2}$
- The number of asymmetric binary relation possible on the set A is $3^{(n^2-n)/2}$
- There are at least 2^n transitive relations (lower bound) and at most $2^{n^2} - 2^{\frac{n^2-n}{2}} + 1$ (upper bound)

5. OPERATION ON RELATIONS:

1. $R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \notin R_2\}$

2. $R_2 - R_1 = \{(a,b) \mid (a,b) \in R_2 \text{ and } (a,b) \notin R_1\}$

3. $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$

4. $R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$

PROPERTIES

1) If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.

2) If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $R_1 \cup R_2 = \{(1,2),(2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

3) If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.

4) If R_1 and R_2 are equivalence relations on A,

- $R_1 - R_2$ is not an equivalence relation (reflexivity fails).
- $R_1 - R_2$ is not a partial order (since $R_1 - R_2$ is not reflexive).
- $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$ is neither equivalence relation nor partial order (reflexivity fails)

5) The union of two equivalence relation on a set is not necessarily an equivalence reation on the set.

6) The inverse of a equivalence relation R is an equivalence relation.

6. COMPOSITON OF RELATIONS

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1, denoted as $R_1 \; R_2$ or simply R_1R_2 is

$$R_1 \; R_2 = \{(a,c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a,b) \in R_1, (b,c) \in R_2)\}$$

NOTE

$R_1 (R_2 \cap R_3) \subset R_1 R_2 \cap R_1 R_3$

$R_1 (R_2 \cup R_3) = R_1 R_2 \cup R_1 R_3$

$R_1 \subseteq A \times B, R_2 \subseteq B \times C, R_3 \subseteq C \times D. (R_1R_2)R_3 = R_1(R_2R_3)$

$(R_1 o R_2)^{-1} = R_2^{-1} o R_1^{-1}$

7. EQUIVALENCE CLASS

Equivalence class of $a \in A$ is defined as $[a] = \{x \mid (x,a) \in R\}$, that is all the elements related to a under the relation R.

Example

E=Even integers, O=odd integers.

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = E \cup O$

The subset E is called the equivalence class containing zero and is denoted by [0].

Properties: consider an equivalence relation R defiend on a set A.

1. $\bigcup_{\forall a \in A} [a] = A$

2. For every a,b∈A such that $a \in [b], a \neq b$ it follows that $[a] = [b]$

3. $\sum_{\forall x \in A} |[x]| = |R|$

4. For any two equivalence class $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \phi$

5. For all $a, b \in A$, if $a \in [b]$ then $b \in [a]$

6. For all $a, b, c \in A$, if $a \in [b]$ and $b \in [c]$, then $a \in [c]$

7. For all $a \in A, [a] \neq \phi$

Congruence modulo n given by $a \equiv b \pmod n$ if and only if n divides $(a - b)$.

8. BINARY OPERATIONS

Let S be a non-empty set. A function $f : S \times S \to S$ is called a binary opertion on set S.

Note

Number of binary operations on a set containing n elements is n^{n^2}

"IN MATHEMATICS
THE ART OF PROPOSING A QUESTION
MUST BE HELD OF HIGHER VALUE THAN SOLVING IT."

– *Georg Cantor*

FUNCTION

1 Classification of function

01. Constant function

$f(x) = k$, k is a constant.

02. Identity function

The function $y = f(x) = x$, $\forall x \in R$

Here domain & Range both R

03. Polynomial function

$y = f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$, n is non negative integer, a_i are real constants. Given $a_0 \neq 0$, n is the degree of polynomial function

There are two polynomial functions, $f(x) = 1 + x^n$ & $f(x) = 1 - x^n$ satisfying the relation: $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ where 'n' is a positive integer.

4. Rational functions

It is defined as the ratio of two polynomials.

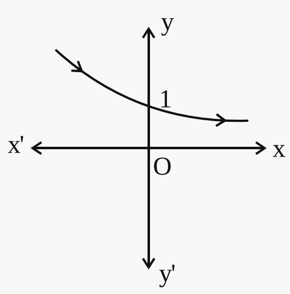
$f(x) = \frac{P(x)}{Q(x)}$ provided $Q(x) \neq 0$

Dom $\{f(x)\}$ is all real numbers except when denominator is zero $[i, e, Q(x) \neq 0]$

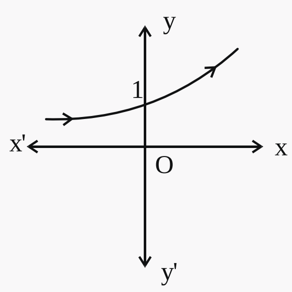
2 Exponential function

$f(x) = a^x, a > 0, a \neq 1.$

0<a<1



a > 1

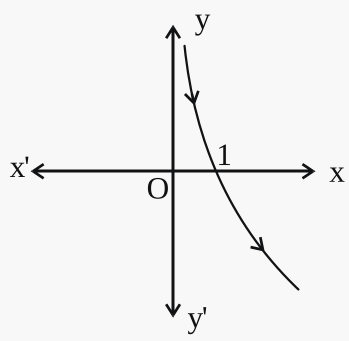


Domain =R, Range = (0, ∞)

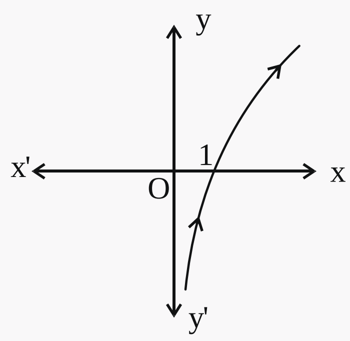
3 Logarithmic function

$f(x) = \log_a x\ [a > 0, a \neq 1]$

0 < a < 1



a > 1



Domain =(0, ∞), Range =R

Proprieties of Log. Functions

1. $\log_a(xy) = \log_a|x| + \log_a|y|$, where $a > 0, a \neq 1$ and $xy > 0$

2. $\log_a x = \frac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$

3. $\log_a\left(\frac{x}{y}\right) = \log_a|x| - \log_a|y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

4. $\log_a(x^n) = n\log_a|x|$, where $a > 0, a \neq 1$ and $x^n > 0$

5. $\log_{a^n} x^m = \frac{m}{n}\log_{|a|}|x|$, where $a > 0, a \neq 1$ and $x > 0$

6. $x^{\log_a y} = y^{\log_a x}$ where $x > 0, y > 0, a > 0, a \neq 1$

7. If a>1, then the values of f(x)= $\log_a x$ increase with the increase in x. i.e. $x < y \Leftrightarrow \log_a x < \log_a y$

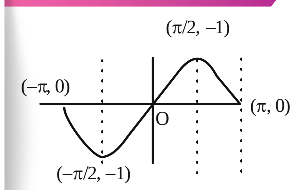
   Also, $\log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$

8. If , 0<a<1, then the values of f (x) = $\log_a x$ decrease with the increase in x . i.e. $x < y \Leftrightarrow \log_a x > \log_a y$

   Also, $\log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

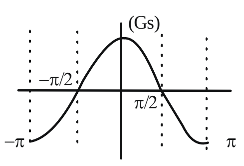
4 Trigonometric Functions

Sine function



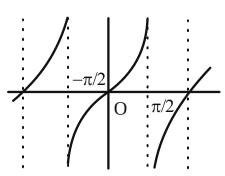
f(x) = sin x.
Dom (f) = R
Ran (f) = [–1, 1]

Cosine function



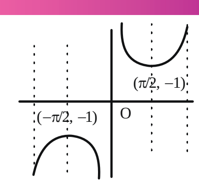
f(x) = cos x
Dom (f) = R
Ran (f) = [-1, 1].

Tangent function



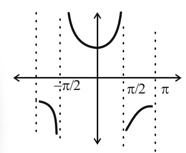
$f(x) = \tan x$
$\text{Dom}(f) = R - \left\{\frac{(2n+1)\pi}{2}, n \in z\right\}$
$\text{Ran}(f) = R$

Cosecant Function



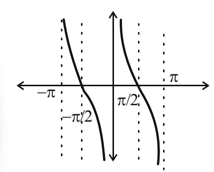
f(x) = cosec x
Dom (f) = R–{mR, n e z }
Ran (f) = R–(–1, 1)

Secant Function



f(x)=sec x
Dom (s) = R - $\{(2n+1)\frac{\pi}{2} | n \in Z\}$
Ran (f) = R - (-1, 1).

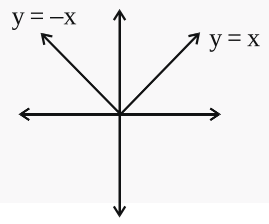
Cotangent function



f(x) = cot (x)
Dom (f) = R –{nπ | n ∈ Z }
Ran (f) = R.

5 Absolute Value Function

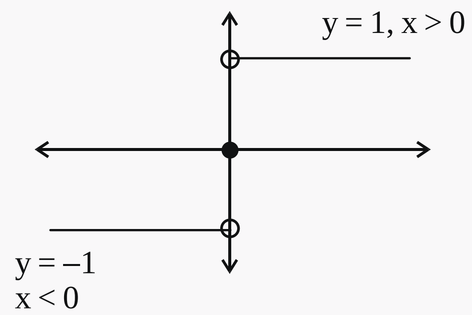
$$y = f(x) = |x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$



1. $|x|^2 = x^2$
2. $\sqrt{x^2} = |x|$
3. $|x| = \max\{-x, x\}$
4. $-|x| = \min\{-x, x\}$
5. $\max(a,b) = \frac{a+b}{2} + \left|\frac{a-b}{2}\right|$
6. $\min(a,b) = \frac{a+b}{2} - \left|\frac{a-b}{2}\right|$
7. $|x+y| \leq |x| + |y|$
8. $|x+y| = |x| + |y|$ if $xy > 0$
9. $|x-y| = |x| + |y|$ if $xy \leqslant 0$
10. $|x| \geqslant a$ (is – ve) $x \in R$
11. $a \leqslant |x| \leqslant b \Rightarrow -b \leq x \leq -a$ or $a \leqslant x \leqslant b$. $x \in [-b, -a] \cup [a, b]$.

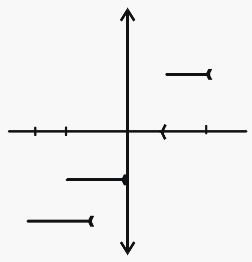
6 Signum Function

$$y = \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



7 Greatest Integer Function

f(x) = [x] the integral part of x, which is nearest & smaller integer



1. $[x] \leqslant x < [x] + 1$
2. $x - 1 < [x] < x$
3. $I \leq x < I + 1 \Rightarrow [x] = I$
4. $[x] - [-x] = \begin{cases} 2x & , x \in I \\ 2x+1 & , 2 \notin I. \end{cases}$
5. $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \in I, \end{cases} \left| \begin{matrix} 2x, & x \in I \\ 2x+1, & , x \notin I \end{matrix} \right.$
6. $[x] \leq n \Leftrightarrow x < n+1, n \in I$
7. $[x] < n \Leftrightarrow x < n$
8. $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$
9. $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \cdots = n$
10. $[x] + [y] \leqslant [x+y] \leqslant [x] + [y] + 1$
11. $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$

8 Fractional Part Function

: y={x} fractional past of x.

y = {x} = x - [x]

1. {x} = x, 0 ≤ x < 1.
2. {x} = 0, x ∈ I
3. {-x} = 1-{x}, x ∉ I
4. {x ± integer} = {x}

9 Odd and Even Function

1. if f(-x) =−f(x) ∀ x ∈R then f is an odd function, odd functions are symmetrical in opposite quadrants or about origin.

2. If f(-x) = f(x). then even. It is symmetric about y axis.

Properties

1. Product of two odd or two even function is an even function.
2. Product of odd & even function is an odd function.
3. Every function can be expressed as the sum of an even and odd function, i.e,

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$$

4. Derivative of an odd function is an even function and of an even is odd.

10 Periodic function

f(x) is periodic if f(x + T) = f(x) $\forall$ x $\in$ R , T = period

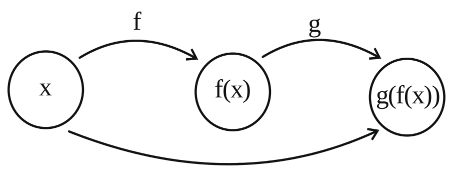
Functions	Period
$\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$	π(n is even), 2π(n odd/ fraction)
$\tan^n x, \cot^n x$	π
|trig function|	π
$x-[x]$	1
$f(x)=$ constant	Periodic with no fundamental period.

Properties of Periodic functions

If f(x) is periodic with period T, then

1. c. f(x) is periodic with period T
2. f(x + c) is periodic with period T.
3. f(x) ± c is periodic with period T.
4. $kf(cx+d)$ has period $\frac{T}{|c|}$ period is only affected by coefficient of x.

11 Composition of Function



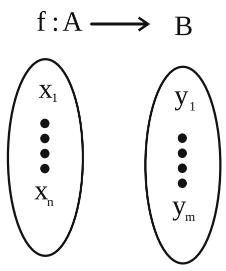
- h(x) = g f(x) = (gof) (x).
- gof ≠ fog.
- Composition of two bijection is a bijection.

Properties of Composite Function

f	g	fog
even	even	even
odd	odd	odd
even	odd	ever
odd	even	even

12 Kinds of Mapping

1. **One-one/Injective/Homomorphic**: f(x) = f(y) ⇒ x=y, then one-one. Graphically, if no line parallel to x-axis meets the graph of function at more than one point.
2. **Onto/Surjective**: If range = co-domain. Method to show subjectivity: Finding the range of y = f(x) & Showing range of f = c σ-domain of f
3. **Many-one mapping** :If two or more element in domain have same image in co-domain.
4. **Into Function** :There's an element in B not having a pre image in A under f. [f : A → B].



13 Number of functions

Total no of functions $= m^n$

Number of One to one functions $=\begin{cases} {}^mP_n, & m\ge n \\ 0, & m<n \end{cases}$

No. of many one functions $=\begin{cases} m^n-{}^mP_n, & m\geqslant n \\ m^n, & m<n \end{cases}$

No. of constant function $= m$

No. of onto function $=\begin{cases} \sum_{r=0}^{n}(-1)^r\,{}^mC_r\,(m-r)^n, & n\ge m \\ m^n, & n<m \end{cases}$

No. of one-to-one onto functions = n!, if m= n

14 Inverse of a function

g:B→A , f(x) = y ⇔ g(y) = x $\forall$ x $\in$ A and y $\in$ B.
Then g is inverse of f

1. Inverse of a bijection is unique.

2. If f : A → B is a bijection g : B →A is the inverse of f, then fog = I_B gof = I_A, where I_A & I_B are identity function

3. The inverse of a bijection is also a bijection $(gof)^{-1} = f^{-1} \circ g^{-1}$.

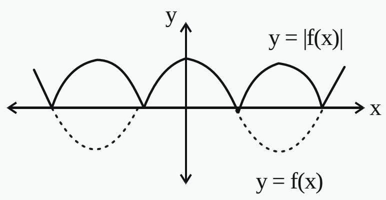
15 Hyperbolic functions

$\sinh x=\frac{e^x-e^{-x}}{2}$ | $\cosh x=\frac{e^x+e^{-x}}{2}$ | $\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ | $\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$

16 Elementary transformation of graphs

01

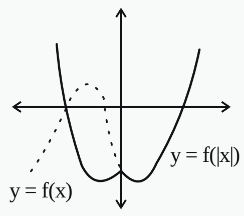
Drawing the graph of y = |f(x)| from the



$$|f(x)| = \begin{cases} f(x) & \text{if} \quad f(x) \geqslant 0 \\ -f(x) & \text{if} \quad f(x) < 0. \end{cases}$$

02

Drawing graph of y=f(|x|) from the known graph of y=f(x).

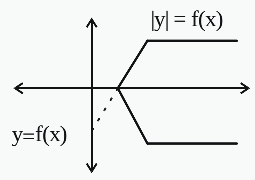


$$f(|x|) = \begin{cases} f(x), & x \geqslant 0 \\ f(-x), & x < 0 \end{cases}$$

Neglect the curve for x<0 & take the images of curve for x⩾0 about y axis.

03

Drawing graph of |y|=f(x) from the know m graph of y=f(x).



Remove portion that lies below x axis. Plot the remaining portion of the graph & also its mirror image in x-axis.

17 Things to remember

Range of $a\cos x + b\sin x$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

Range of $f(x) = \sqrt{a-x} + \sqrt{x-b}$ if $a > b > 0$ is $\sqrt{a-b} + \sqrt{2(a-b)}$

Range of $\left(x+\frac{1}{x}\right) + \frac{1}{\left(x+\frac{1}{x}\right)}$ is $(-\infty, -2.5] \cup [2.5, \infty)$

$-\sin 1 < \sin(\cos x) < \sin 1$

$\cos 1 < \cos(\sin x) < 1$

18 Functional Equation

1) $f(x+y) = f(x)f(y)$, then $f(x) = a^x$

2) $f(xy) = f(x) + f(y)$, then $f(x) = \log_a x$

3) $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$, then $f(x) = mx + c$

4) $f(x)f\left(\frac{1}{x}\right) = 1$, then $f(x) = \pm x^n$

Inverse Trigonometric Functions

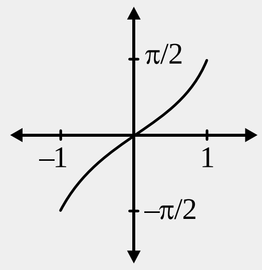
01

Inverse function	Domain	Principal Value Branch
$y = \sin^{-1}x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1}x$	$R-(-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$y = \sec^{-1}x$	$R-(-1,1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$y = \tan^{-1}x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	R	$(0, \pi)$

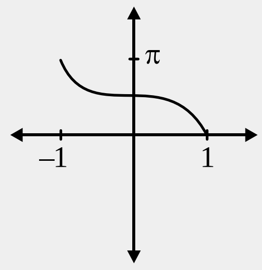
02

Graph

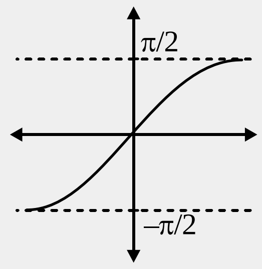
1. $y = \sin^{-1}x$



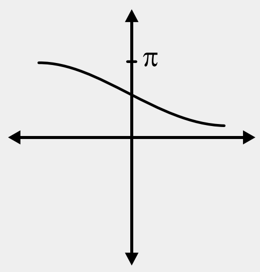
2. $y = \cos^{-1}x$



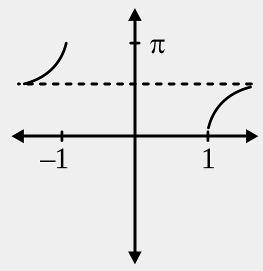
3. $y = \tan^{-1}x$



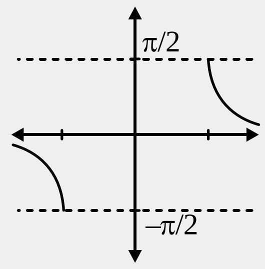
4. $y = \cot^{-1}x$



5. $y = \sec^{-1}x$



6. $y = \text{cosec}^{-1}x$



Properties Of Inverse Trigonometric Functions

03

Property -01

(i) $\sin^{-1}(\sin\theta) = \theta$ if $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos\theta) = \theta$ if $0 \le \theta \le \pi$

(iii) $\tan^{-1}(\tan\theta) = \theta$ if $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot\theta) = \theta$ if $0 < \theta < \pi$

(v) $\sec^{-1}(\sec\theta) = \theta$ if $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, if $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

Property -02

(i) $\sin\left(\sin^{-1}x\right) = x$, if $-1 \le x \le 1$

(ii) $\cos\left(\cos^{-1}x\right) = x$, if $-1 \le x \le 1$

(iii) $\tan\left(\tan^{-1}x\right) = x$, if $-\infty < x < \infty$

(v) $\sec\left(\sec^{-1}x\right) = x$, if $-\infty < x \le -1$ or $1 \le x < \infty$

(iv) $\cot\left(\cot^{-1}x\right) = x$, if $-\infty < x < \infty$

(vi) $\text{cosec}\left(\text{cosec}^{-1}x\right) = x$, if $-\infty < x \le -1$ or $1 \le x < \infty$

Property -03

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, if $-1 \le x \le 1$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, if $-1 \le x \le 1$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$, if $-\infty < x < \infty$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $-\infty < x < \infty$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $-\infty < x \le -1$ or $1 \le x < \infty$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, if $-\infty < x \le -1$ or $1 \le x < \infty$

Property -04

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1,1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, x \in (-\infty,-1] \cup [1,\infty)$

Property -05

(i) $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1 - \{0\}$

(ii) $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right), x \in R - (-1,1)$

(iii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right), -1 \le x \le 1 - \{0\}$

(iv) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), x \in R - (-1,1)$

(v) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right), x \in R - \{0\}$

(vi) $\tan^{-1}(1/x) = \begin{cases} \cot^{-1}x & \forall \ x > 0 \\ -\pi + \cot^{-1}x & \forall \ x < 0 \end{cases}$

Property -06

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), x > 0, y > 0, xy > 1$

$= -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), x < 0, y < 0, xy > 1$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

$= \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), x > 0, y > 0, xy < -1$

$= -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, *if* $x < 0, y < 0$ *and* $xy > 1$

(iii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$, if $x > 0, y > 0, z > 0$ and $(xy+yz+zx) < 1$, *if* $x < 0, y < 0$ *and* $xy > 1$

Property -07

(i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$, *if* $x^2+y^2\le 1$ *or if* $xy<0$ *and* $x^2+y^2>1$; *where* $x,y\in[-1,1]$

$=\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$ if $0<x\le 1, 0<y\le 1$ and $x^2+y^2>1$

$=-\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$, *if* $-1\le x<0, 0<y\le 1$ *and* $x^2+y^2>1$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $xy>0$, $x^2+y^2>1$ or $x^2+y^2\le 1$

$=\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $0<x\le 1$, $-1\le y\le 0$, $x^2+y^2>1$

$=-\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, $-1\le x<0$, $0<y\le 1$, $x^2+y^2>1$

Property -08

(i) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1$, $x+y\ge 0$

$=2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1$, $x+y<0$

(ii) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le x, y\le 1$, $x\le y$

$=-\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, $-1\le y\le 0$, $0<x\le 1$, $x>y$

Property -09

(i) $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right), \frac{-1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

$=\pi-\sin^{-1}\left(2x\sqrt{1-2^2}\right), \quad \frac{1}{\sqrt{2}}\le x\le 1.$

$=-\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), -1\le x\le\frac{-1}{\sqrt{2}}$

(ii) $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right), -1/2\le x\le 1/2$

$=\pi-\sin^{-1}\left(3x-4x^3\right), \ 1/2<x\le 1$

$=-\pi-\mathrm{san}^{-1}\left(3x-4x^3\right), -1\le x<-1/2$

Property -10

(i) $2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right), \quad 0\le x\le 1. =2\pi-\cos^{-1}\left(2x^2-1\right), -1\le x\le 0$

(ii) $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right), \quad 1/2\le x\le 1$

$=2\pi-\cos^{-1}\left(4x^3-3x\right), -1/2\le x\le 1/2$

$=2\pi+\cos^{-1}\left(4x^3-3x\right), -1\le x\le -1/2$

Property -11

(i) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad -1<x<1$

$=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad x>1$

$=-\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad x<-1$

(ii) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right), \quad -1\le x\le 1$

$=\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \ x>1$

$=-\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \ x<-1$

(iv) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ \frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

$=\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ x>\frac{1}{\sqrt{3}}$

$=-\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ x<\frac{-1}{\sqrt{3}}$

(iii) $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \quad 0\le x<\infty$

$=-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), -\infty<x\le 0$

Property -12

(i) $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\mathrm{cosec}^{-1}\left(\frac{1}{x}\right), x>0$

$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\cot^{-1}\frac{x}{\sqrt{1-x^2}}=\sec^{-1}\left(\frac{1}{x}\right)=\mathrm{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), x>0$

(iii) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\mathrm{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

Property -13

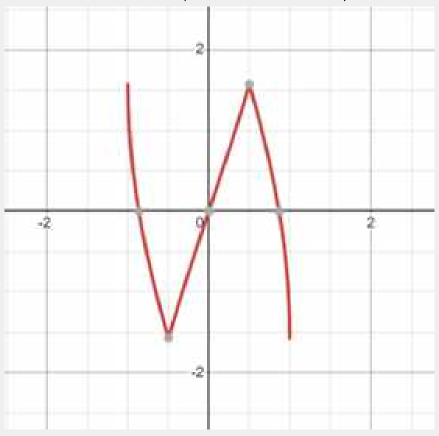
If $x_1, x_2, \ldots\ldots, x_n \in R$ then $\tan^{-1}x_1+\tan^{-1}x_2+\ldots\ldots+\tan^{-1}x_n=$

$$\tan^{-1}\left(\frac{s_1-s_3+s_5-s_7+\cdots}{1-s_2+s_4-s_6+\cdots}\right)$$

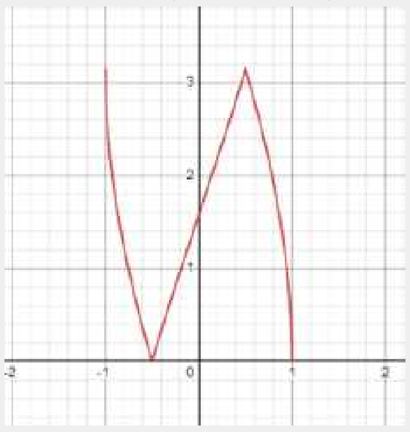
where, s_k = sum of products of $x_1, x_2, \ldots\ldots, k_n$ taken k at a time.

Some Important graphs

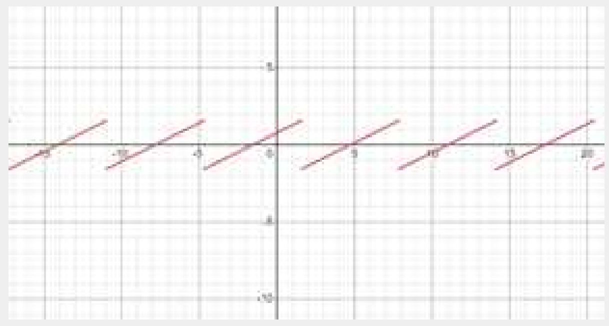
$\sin^{-1}\left(3x-4x^3\right)$



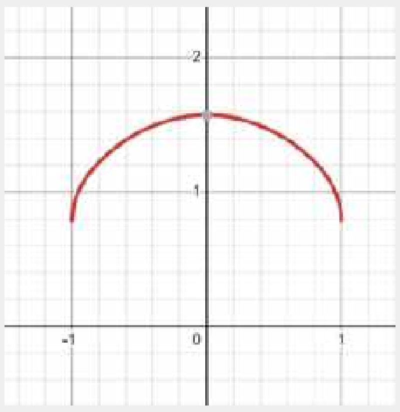
$\cos^{-1}\left(4x^3-3x\right)$



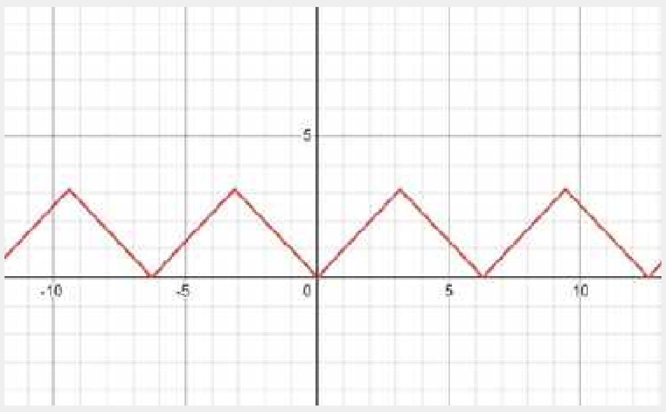
$\tan^{-1}\left(\sec x+\tan x\right)$



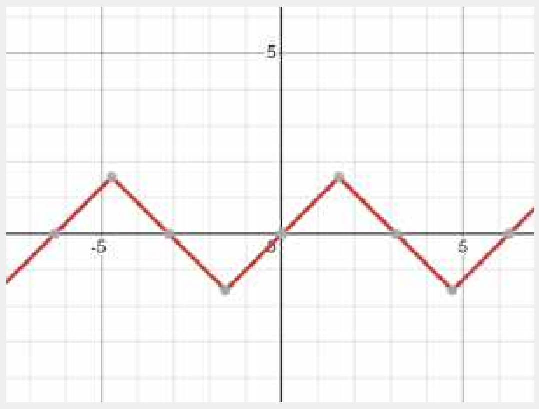
$\tan^{-1}\left(\frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$



$\cos^{-1}\left(\cos x\right)$



$\sin^{-1}\left(\sin x\right)$



MATRICES

01. MATRIX

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix

02. ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order m×n or simply m×n matrix.

or $A=[a_{ij}]_{m\times n}$, $1\le i\le m$, $1\le j\le n$, $i, j \in N$

a_{ij} is an element lying in the i^{th} row & j^{th} column. The number of elements in m×n matrix will be mn.

03. TYPE OF MATRIX

(i) Column Matrix: A matrix is said to be a column matrix if it has only one column, i.e., $A=[a_{ij}]_{m\times 1}$ is a column matrix of order m×1.

(ii) Row Matrix: Row matrix has only one row, i.e., $B=[b_{ij}]_{1\times n}$ is a row matrix of order 1×n.

(iii) Square Matrix: Square matrix has equal number of rows and columns, i.e., $A=[a_{ij}]_{m\times m}$ is a square matrix of order m.

(iv) Diagonal Matrix: A square matrix is said to be diagonal matrix if all of its non-diagonal elements are zero, i.e., $B=[b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij}=0$, where $i \ne j$.

(v) Scalar Matrix: It is a diagonal matrix with all its diagonal elements are equal, i.e., $B=[b_{ij}]_{m\times n}$ is a scalar matrix if $b_{ij} = 0$, where $i\neq j$, $b_{ij}=k$, when I = j & k= constant.

(vi) Identity Matrix: : It is a diagonal matrix having all its diagonal elements equal to 1, i.e., $A=[a_{ij}]_{m\times n}$ is an identity matrix if

$$a_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\ne j\end{cases}$$

we denote identity matrix by I_n when order is n.

(vii) Zero Matrix: A matrix is said to be zero or null matrix if all its elements are zero. It is denoted by O.

04. EQUALITY OF MATRICES

Two matrices $A=[a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for all i & j

05. TRACE OF A MATRIX

The sum of diagonal element of a square matrix A is called the trace of matrix A, which is denoted by tr A

$$\text{tr } A=\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots a_{nn}$$

Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]$ and λ be a scalar.

(i) $tr(\lambda A) = \lambda tr(A)$ (ii) $tr(A - B)= tr(A) - tr(B)$

(iii) $tr(AB) = tr(BA)$ (iv) $tr(I_n)= n$

(vi) $tr(AB) \neq tr A. tr B$ (v) $tr(O) = 0$

06. ADDITION OF MATRICES

Properties of matrix Addition

(i) Commulative Law: A + B = B + A **(ii) Associative Law:** (A + B) + C = A + (B + C)

(iii) Existence of Additive Identity: Let $A=[a_{ij}]_{m\times n}$ & O = zero matrix of order m × n, then A + O = O + A = A. Here O is the additive identity for matrix addition.

(iv) Existence of Additive Inverse: Let $A= [a_{ij}]_{m\times n}$ be any matrix then we have another matrix as Let $-A= [-a_{ij}]_{m\times n}$ such that A + (-A) = (- A) + A = O. Here -A is the additive inverse of A or negative of A.

07. MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A=[a_{ij}]m\times n$ be a matrix & k, t be a number.

Then, $kA = Ak = [ka_{ij}]_{m\times n}$

Properties

(I) k (A + B) = kA + kB **(ii)** (k + t) A = kA + tA.

08. MULTIPLICATION OF MATRICES

If A & B are any two matrices, then their product will be defined only when the number of columns in A is equal to the number of rows in B

If $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{n\times p}$ then their product $AB=C=\left[c_{ij}\right]$ is a matrix of order, m x p where

$(ij)^{th}$ element of $AB=C_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}$

09. PROPERTIES OF MATRIX MULTIPLICATON

(I) Associative Law for Multiplication: If A, B & C are three matrices of order $m\times n, n\times p \& p\times q$ respectively, then $(AB)C = A(BC)$

(ii) Distributive Law : For three matrices A, B & C (a) $A(B+C) = AB + AC$
(b) $(A+B)C = AC + BC$ whenever both sides of equality are defined.

(iii) Matrix Multiplication is not commutative in general, i.e. $AB \neq BA$ (in general).

(iv) Existence of Multiplicative Identity: For every square matrix, there exists an identity matrix I of same order such that $IA = AI = A$

10. PROPERTIES OF TRANSPOSE OF THE MATRICES

For any matrices A & B of suitable orders, we have:

(i) $(A^T)^T = A$

(ii) $(kA)^T = k\,(A)^T$ (where k is constant)

(iii) $(A \pm B)^T = A^T \pm B^T$

(iv) $(AB)^T = B^T A^T$

(v) $(A_1 A_2 A_3 \ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$

(vi) $I^T = I$.

11. MATRIX POLYNOMIAL

Let $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots a_{n-1} x + a_n$ be a polynomial and let A be a square matrix of order n, then $f(A) = a_0 A^m + a_1 A^{m-1} + a_2 A^{m-2} + \ldots + a_{n-1} A + a_n I_n$ is called a matrix polynomial.

12. SYMMETRIC & SKEW SYMMETIC MATRICES

Symmetric Matrix

A square matrix $\mathbf{A} = [a_{ij}]$ is called a symmetric matrix, if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a skew-symmetric matrix, if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

Properties of Symmetric & Skew Symmetric Matrices

(i) For any square matrix A with real number entries $(A + A^T)$ is a skew symmetric matrix $(A - \mathbf{A}^T)$ symmetric matrix.

(ii) Any square matrix A can be expressed as the sum of a symmetric & a skew symmetric matrix as $A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$

13. INVERTIBLE MATRIX AND INVERSE MATRIX

Properties of Invertible Matrices

(i) Uniqueness of Inverse : Inverse of a square matrix, if it exists, is unique.

(i) $(A^{-1})^{-1} = A$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(AB)^{-1} = B^{-1}A^{-1}$ (iv) $(A^k)^{-1} = (A^{-1})^k$

14. ORTHOGONAL MATRIX

A square matrix A is called orthogonal if $AA^T = I = A^T A$, i.e., if $A^{-1} = A^T$.

Example: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$. In fact every unit matrix is orthogonal

15. IDEMPOTENT MATRIX

A square matrix A is called an idempotent matrix if $A^2 = A$.

Example: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because

$$A^2 = \begin{bmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ and $B^2 = B$.

16. INVOLUTORY MATRIX

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ In fact every unit matrix is involutory

17. PERIODIC MATRIX

A matrix A will be called a periodic matrix if $A^{k+1} = A$ where k is a positive integer. If, however k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A.

18. NILPOTENT MATRIX

A square matrix A is called a nilpotent matrix if there exists $p \in N$ such that $A^p = 0$.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ (Here p = 2)

19. DIFFERENTIATION OF A MATRIX

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$, then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A.

DETERMINANT

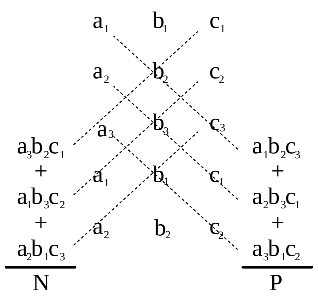
1. DETERMINANT OF A SQUARE MATRIX OF ORDER TWO AND THREE

Expansion of two order: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

Expansion of three order: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

2. SARRUS RULE

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$\Rightarrow \Delta = P - N$

3. PROPERTIES OF DETERMINANTS

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

(iii) If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

(v) If some or all the elements of a row or column of a determinant are expressed as a sum of two (or more) terms, then the determinant can be expressed as a sum of two (or more) determinants.

(vi) If the equimultiples of corresponding elements of other row (or column) are added to each element of any row or column of a determinant, then the value of the determinant remains the same.

(vii) $|A^T| = |A|$, where A^T= transpose of A

(viii) If $A = [a_{ij}]_{3\times3}$, then $|kA| = k^3 |A|$.

(ix) The determinant of the product of matrices is equal to product of their respective determinants, i.e., $|AB|=|A||B|$, where A& B are square matrices of same order

(x) $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3$

4. USE OF DETERMINANTS IN CO-ORDINATE GEOMETRY

(i) Area of triangle, whose vertices are (x_r, y_r) $\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

(ii) If $a_rx + b_ry + c_r = 0$ are the sides of a triangle, then the area =

$$\left|\frac{1}{2c_1c_2c_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2\right|$$

c_1, c_2, c_3 are cofactors of c_1, c_2, c_3.

(iii) Equation of a straight line passing through two points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iv) If three lines $a_rx + b_ry + c_r = 0$ are concurrent, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(v) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight line then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(vi) The equation of circle through three non-collinear points (x_r, y_r) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

5. MINOR AND COFACTOR OF AN ELEMENT OF A DETERMINANT

Minor: The determinant that is left by cancelling the row and column intersecting at a particular element of a determinant is called the minor of that element of the determinant. Minor of an element a_{ij} of a determinant is denoted by M_{ij}.

Cofactor: The cofactor of an element a_{ij} of a determinant is denoted by A_{ij} (or Cij) and is equal to $(-1)^{i+j}\ M_{ij}$.

6. ADJOINT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\text{adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of a_{ij}

If A be any given square matrix of order n, then A(adj A)= (adj A A=|A|I where I is the identity matrix of order n.

7. PROPERTIES OF ADJOINT OF A MATRIX

If A be any given square matrix of order n, we can define the following:

- A(adj A) = adj A) A = AI, where I is the identity matrix of order n
- For a zero matrix 0, adj(0) = 0
- For an identity matrix I, adj(I) = I
- For any scalar k, $\text{adj}(kA) = k^{n-1}\text{adj}(A)$
- $\text{adj}(A^T) = (\text{adj } A)^T$
- det(adj A), i.e. $|\text{adj } A| = (\det A)^{n-1}$
- If A is an invertible matrix and A^{-1} be its inverse, then : adj A = $(\det A)^{A-1}$ is invertible with inverse $(\det A)^{-1}$ $A\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

  Suppose A and B are two matrices of order n, then adj(AB) = (adj B)(adj A)
- For any non-negative integer p, $\text{adj}(A^p) = (\text{adj } A)^p$

  If A is invertible, then the above formula also holds for negative k.

8. SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be singular if |A| = 0, otherwise it is called non-singular matrix.

If A & B are non-singular matrix of same order, then AB & BA are also non-singular matrices of same order.

9. INVERSE OF A MATRIX

If A and B are two matrices such that AB=I=BA

then B is called the inverse of A and it is denoted by A^{-1}

Also, $A^{-1} = \frac{\text{adj}\,A}{|\mathbf{A}|}$, if $|A| \neq 0$

10. PROPERTIES OF INVERSE

If A and B are the non-singular matrices, then the inverse matrix should have the following properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \ldots A_2^{-1}A_1^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(kA)^{-1} = (1/k)A^{-1}$
- $AB = I_n$, where A and B are inverse of each other.

11. PRODUCT OF TWO DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

Then product of Δ_1 and Δ_2 is defined as

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1x_1+a_2x_2+a_3x_3 & a_1y_1+a_2y_2+a_3y_3 & a_1z_1+a_2z_2+a_3z_3 \\ b_1x_1+b_2x_2+b_3x_3 & b_1y_1+b_2y_2+b_3y_3 & b_1z_1+b_2z_2+b_3z_3 \\ c_1x_1+c_2x_2+c_3x_3 & c_1y_1+c_2y_2+c_3y_3 & c_1z_1+c_2z_2+c_3z_3 \end{vmatrix}$$

12. DIFFERENTIATION OF A DETERMINANT

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$

then $\Delta'(x)$

$$= \begin{vmatrix} f'_1(x) & g'_1(x) & h'_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f'_2(x) & g'_2(x) & h'_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f'_3(x) & g'_3(x) & h'_3(x) \end{vmatrix}$$

13. INTEGRATION OF DETERMINATION

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$, where p, q, r, l, m and n are constants.

Then $\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ p & q & r \\ l & m & n \end{vmatrix}$

14. USE OF SUMMATION

If $f(r) = \begin{vmatrix} r & r^2 & r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$ where p, q, t are constants, then $\sum_{r=1}^{n} f(r) = \begin{vmatrix} \sum_{r=1}^{n} r & \sum_{r=1}^{n} r^2 & \sum_{r=1}^{n} r^3 \\ p & q & t \\ 1 & 2 & 3 \end{vmatrix}$

15.

$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$

- Consistent system of equation has solution
  - Unique solution: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (Equation represents intersecting lines)
  - Infinite solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Equation represents coincident lines)
- Inconsistent system of equation has no solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Equation represents parallel disjoint lines)

16.

Check value of Δ

- $\Delta \neq 0$: Consistent system and has unique solution $x = \frac{\Delta_1}{\Delta}; y = \frac{\Delta_2}{\Delta}; z = \frac{\Delta_3}{\Delta}$
- $\Delta = 0$: Check the values of Δ_1, Δ_2 and Δ_3
  - At least one of Δ_1, Δ_2 and Δ_3 is not zero: Inconsistent system
  - $\Delta_1 = \Delta_2 = \Delta_3 = 0$: Put z = t and solve any two equations to get the values of x and y in terms of t.

17.

1. Symmetric determinant

The elements situated at equal distance from the diagonal are equal both in magnitude and sign.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

2. Skew symmetric determinant

All the diagonal elements are zero and the elements situated at equal distance form the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

3. Circulant determinant:

The elements of the rows (or columns) are in cyclic arrangement

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

4. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$

5. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

6. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$

LIMITS

1. Right hand and Left hand Limit

To evaluate $\lim_{x \to a^+} f(x)$

1. Put x = a + h in f (x) to get $\lim_{h \to 0} f(a+h)$
2. Take the limit as h $\to$ 0.

To evaluate $\lim_{x \to a^-} f(x)$.

1. Put x = a − h in f (x) to get $\lim_{h \to 0} f(a-h)$.
2. Take the limit as h $\to$ 0.

2. Some Useful Limits

(1) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where n $\in$ Q, the set of rational numbers.

(2)

(i) $\lim_{n \to \infty} a^n = \begin{cases} \infty, & if\ a > 1 \\ 1, & if\ a = 1 \\ 0, & if\ -1 < a < 1 \\ does\ not\ exist, & if\ a \le -1 \end{cases}$

(ii) $\lim_{x \to \infty} \frac{a_0 x^p + a_1 x^{p-1} + \ldots\ldots + a_{p-1} x + a_p}{b_0 x^q + b_1 x^{q-1} + \ldots\ldots + b_{q-1} x + b_q}$

$= \begin{cases} \frac{a_0}{b_0}, & if\ p = q \\ 0, & if\ p < q \\ \infty, \frac{a_0}{b_0} > 0 & if\ p > q \end{cases}$

(3) Trigonometric Limits

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \to 0} \frac{\cos x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(v) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(vi) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$.

Some Useful Expansions

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ to ∞

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ to ∞

(iii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots$ to ∞

(iv) $\sin^{-1} x = x$

$+ \frac{1^2.x^3}{3!} + \frac{1^2.3^2}{5!} X^5 + \frac{1^2.3^2.5^2}{7!} x^5 + \ldots$ to ∞

(v) $(\sin^{-1} x)^2$

$= \frac{2}{2!} x^2 + \frac{2.2^2}{4!} x^4 + \frac{2.2^2.4^2}{6!} x^6 + \ldots$ to ∞

(vi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ to ∞.

(4) Exponential and Logarithmic Limits

(i) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$, a > 0

(iii) $\lim_{x \to 0} \frac{a^x - b^x}{x} = \log_e \left(\frac{a}{b}\right)$; a, b > 0

(iv) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$

(v) $\lim_{n \to 0} \left(1 + \frac{1}{n}\right)^n = e$

(vi) $\lim_{h \to 0} (1 + ah)^{1/h} = e^a$

(vii) $\lim_{x \to \infty} \frac{\log x}{x^m} = 0$, (m > 0)

(viii) $\lim_{x \to 0} \frac{\log_a (1+x)}{x} = \log_a e$, (a > 0, a ≠ 1)

(ix) $\lim_{x \to 0} \left(1 + \frac{a}{x}\right)^x = e^a$

(x) $\lim_{x \to 0} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e$, where f (x) $\to \infty$ as x $\to$ 0.

(xi) $\lim_{x \to 0} (1 + f(x))^{1/f(x)} = e$, where f(x) $\to$ 0 as x $\to$ 0

Some Useful Expansions

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

(ii) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ to ∞

(iii) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ to ∞, $-1 < x \le 1$

(iv) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$ to ∞, $-1 \le x < 1$

(v) $a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2!} + \ldots$ to ∞.

(vi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} X^2 + \ldots$ to ∞, $-1 < x < 1$,

Note

(i) If $\lim_{x \to a} f(x) = A > 0$ and $\lim_{x \to a} g(x) = B$, then $\lim_{x \to a} [f(x)]^{g(x)} = A^B$.

(ii) If $\lim_{\to 0} f(x) = 1$ and $\lim_{x \to 0} g(x) = \infty$, then

$\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$.

3. EVALUATION OF LIMITS USING L'HOSPITAL'S RULE

(i) $\left(\frac{0}{0}\right)$ form: If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

(ii) $\left(\frac{\infty}{\infty}\right)$ form: If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

Note: That sometimes we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.

CONTINUITY AND DIFFERENTIABILITY

01 Continuity

Continuity of a Function at a Point

Suppose f is a real function on a subset of the real numbers & let c be a point in the domain of f. Then f is continuous at c if

$\lim_{x \to c} f(x) = f(c)$

Continuity of a Function in an Interval

Suppose f is a function defined on a closed interval [a, b], then for f to be continuous, it needs to be continuous at every point in [a, b] including the end points a & b.

Continuity of f at a, $\lim_{x \to a^+} f(x) = f(a)$

Continuity of f at b, $\lim_{x \to b^-} f(x) = f(b)$

A function which is not continuous at point x=c is said to be discontinuous at that point

02 Algebra of Continuous Functions

Theorem 1: Suppose f & g be two real functions continuous at a real number c, Then

(1) f + g is continuous at x=c

(2) f - g is continuous at x=c

(3) f.g is continuous at x=c

(4) f/g is continuous at x=c, (provided $g(c) \neq 0$)

Theorem 2: Suppose f & g are real valued functions such that (fog) is defined at c. If g is continuous at c& if f is continuous at g(c), then (fog) is continuous at c.

03 Differentiability

A function f is said to be differentiable at a point c in its domain, if its left hand & right hand derivatives exist at c are equal.

Here at x = c,

Left Hand Derivative,

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} = Lf'(c)$$

Right Hand Derivative,

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = Rf'(c)$$

Theorem: If a function f is differentiable at a point c, then it is also continuous at that point. Therefore, every differentiable function is continuous, but the converse is not true.

04 Algebra of Derivatives

Let u, v be the functions of X.

(1) Sum and Difference Rule $(u \pm v)' = u' \pm v'$

(2) Leibnitz or Product Rule $(uv)' = uv' + u'v$

(3) Quotient Rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

05 Chain Rule

If y is a function of u, u is a function of v & v is a function of x.

Then, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

06 Implicit Functions

An equation of the form f(x, y) = 0 in which y is not expressible in terms of x is called an implicit function of x & y.

Derivative of Implicit Functions

Let y=f(x, y), where f(x, y) be an implicit function of x & y.

Firstly differentiate both sides of equation w.r.t x

Then take all terms involving $\frac{dy}{dx}$ on L.H.S. & remaining terms on R.H.S. to get the required value.

07 Differentiation of Inverse Trigonometric Functions

f(x)	f′(x)	Domain of f′
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$	(-1,1)
$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$	(-1,1)
$\tan^{-1}x$	$\frac{1}{1+x^2}$	R
$\cot^{-1}x$	$\frac{-1}{1+x^2}$	R
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert > 1$
$\text{cosec}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\lvert x\rvert > 1$

08 Logarithmic Differentiation

Logarithmic Differentiation is a very useful technique to differentiate functions of the form $f(x)=[u(x)]^{v(x)}$, where f(x) & u(x) are positive.

We apply logarithm (to base) on both sides to the above equation & then differentiate by using chain rule, in this way we can find f' (x). This process is called logarithmic

$$\frac{d}{dx}\left(e^x\right) = e^x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \quad \& \quad \frac{d}{dx}a^x = a^x \log a$$

09 Derivatives of Functions In Parametric Form

The set of equations x = f(t), y = g(t) is called the parametric form of an equation.

Here, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ or $\frac{g(t)}{f(t)}$

Here, $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables.

10 Second Order Derivative

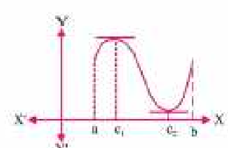
Let y = f(x), then $\frac{dy}{dx} = f'(x)$

If f′(x) is differentiable, then we may differentiate it again w.r.t. x & get the second order derivative represented by:

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ or f″ (x) or D^2y or y″ or y_2

11 Rolle's Theorem

If f: [a, b]→R is continuous on [a, b]& differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f′(c) = 0

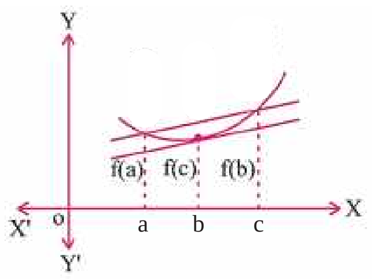


In the above graph, the slope of tangent to the curve at least at one point becomes zero. The slope of tangent at any point on the graph of y = f(x) is nothing but the derivative of f(x) at that point.

12 Mean Value Theorem

If f: [a, b]→R is continuous on [a, b] & differentiable on (a, b). Then there exists some c in (a, b) such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$



The Mean value Theorem states that there is a point c in (a, b) such that the slope of the tangent at (c, f(c)) is same as the slope of the secant between (a, f(a)) and (b, f(b)) or there is a point c in (a, b) such that the tangent at (c, f(c) is parallel to the secant between (a, f(a) & (b, f(b))

Differentiation

1. DERIVATIVE OF A FUNCTION

Derivative at a Point

The value of $f'(x)$ obtained by putting x = a, is called the derivative of $f(x)$ at x = a and it is denoted by $f'(a)$ or $\left\{\frac{dy}{dx}\right\}_{x=a}$.

2. Standard Derivatives

The following formulae can be applied directly for finding the derivative of a function:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(a^x) = a^x \log_e a, a > 1$
9. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$
10. $\frac{d}{dx}(x^n) = nx^{n-1}$
11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
12. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$
14. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
15. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$
16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, -\infty < x < \infty$
17. $\frac{d}{dx}(|x|) = \frac{|x|}{x}$ or $\frac{x}{|x|}, x \neq 0.$

3. RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero, i.e., $\frac{d}{dx}(c) = 0$.
2. The derivative of constant times a function is constant times the derivative of the function, i.e.,

   $\frac{d}{dx}\{c. f(x)\} = c\frac{d}{dx}\{f(x)\}.$
3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

   $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}.$

Note: In general, if $f_1(x), f_2(x), \dots f_n(x)$ are n differentiable functions, then we have $\frac{d}{dx}[f_1(x) \pm f_1(x) \pm \dots \pm f_n(x)] = \frac{d}{dx}[f_1(x)] \pm \frac{d}{dx}[f_2(x)] \pm \dots \pm \frac{d}{dx}[f_n(x)].$

PRODUCT RULE OF DIFFERENTIATION

If $f(x)$ and $g(x)$ are differentiable functions of x, then

$\frac{d}{dx}[f(x)\, g(x)] = f(x)\, g'(x) + g(x)\, f'(x).$

NOTE

$\frac{d}{dx}[f(x)\, g(x)\, h(x)] = f(x)\, g(x)\, h'(x) + f(x)\, g'(x)\, h(x) + f'(x)\, g(x)\, h(x)$

Quotient Rule of Differentiation

If $f(x)$ and $g(x)$ are two differentiable functions of x, then

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$

Differentiation of a Function (Chain Rule)

If y is a differentiable function of u and u is a differentiable function of x, then

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Key Points on Chain Rule

1. The chain rule can be extended further as:
   If y is a function of u, u is a function of v and v is a function of x, then

   $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$ and so on.
2. If $y = u^n$, where u is a function of x, then

   $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nu^{n-1} \times \frac{du}{dx}.$

   $\left[\because \frac{dy}{du} = nu^{n-1}\right]$

4. DERIVATIVE OF PARAMETRIC FUNCTIONS

If x = f (t) and y = g (t), then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}.$$

And $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}$.

5. DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

If y = f (x) and z = g(x), then in order to find the derivative of f (x) w.r.t. g(x), we use the formula

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

6. LOGARITHMIC DIFFERENTIATION

Properties of Logarithms

1. $\log_e (mn) = \log_e m + \log_e n$
2. $\log_e \left(\frac{m}{n}\right) = \log_e m - \log_e n$
3. $\log_e (m)^n = n \log_e |m|$
4. $\log_e e = 1$
5. $\log_n m = \frac{\log_e m}{\log_e n}$
6. $\log_n m . \log_m n = 1.$

Shorter Methods of Finding the Derivative of a Logarithmic Function

If y = $[f(x)]^{g(x)}$, then to find $\frac{dy}{dx}$, in addition to the method discussed above, we can also apply any of the following two methods:

Method 1

Step 1. Express y = $[f(x)]^{g(x)} = e^{g(x) \log f(x)}$

$[\because a^x = e^{x \log a}]$

Step 2. Differentiate w.r.t. x to obtain $\frac{dy}{dx}$

Method 2

Step 1. Evaluate

A = Differential coefficient of y treating f (x) as constant.

Step 2. Evaluate

B = Differential coefficient of y treating g(x) as constant.

Step 3. $\frac{dy}{dx}$ = A + B.

7. DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

Important Substitutions to Reduce the Function to a Simpler Form

Expressions	Substitutions
$\sqrt{a^2 - x^2}$	Put x = a sin θ or x = a cos θ
$\sqrt{x^2 - a^2}$	Put x = a sec θ or x = a cosec θ
$\sqrt{a^2 + x^2}$	Put x = a tan θ or x = a cot θ
$\frac{a-x}{a+x}$ or $\frac{a+x}{a-x}$	Put x = a tan θ
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	Put x = a cos θ
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	Put $x^2 = a^2 \cos θ$

8. DIFFERENTIATION OF A FUNCTION GIVEN IN THE FORM OF A DETERMINANT

If y = $\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note: The differentiation of a determinant can be done in columns also.

APPLICATION OF DERIVATIVES

1. Common application of derivatives.

- Finding Rate of Change of a Quantity
- Finding the Approximation Value
- Finding the equation of a Tangent and Normal To a Curve
- Finding Maxima and Minima, and Point of Inflection
- Determining Increasing and Decreasing Functions

(2) Approximations

Assume we have a function y = f(x), which is defined in the interval [a, a + h], then the average rate of change in the function in the given interval is (f(a + h) − f(a))/h

Now using the definition of derivative, we can write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

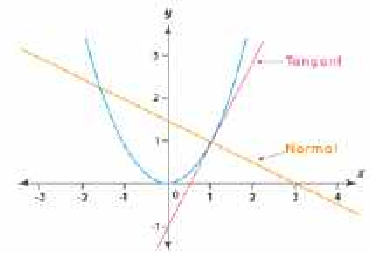
which is also the instantaneous rate of change of the function f(x) at a. Now, for a very small value of h, we can write

$$f'(a) \approx (f(a+h) - f(a))/h$$

(3) Equations of tangent & Normal

- Equation of tangent at (x_1, y_1): $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$
- Equation of Normal at (x_1, y_1):

$$y - y_1 = \left(-\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$



(4) Increasing decreasing functions.

Properties of monotonic functions :

(1) If f(x) is continuous on [a, b] such that f' (c) ≥ 0 for each c, then f(x) is monotonically increasing function. Similar definition goes for monotonically decreasing function.

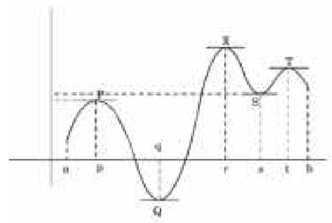
(2) If f(x) is strictly increasing function on [a, b] then $f^{-1}(x)$ exists & is also strictly increasing on [a, b]. Similar result follows for strictly decreasing functions.

(3) If f(x) & g(x) are two continuous & differentiable functions, then we can relate fog (x) & gof (x) by the following table

+ denotes increasing function
− denotes decreasing function

f(x)	g(x)	fog/gof
+	+	+
+	−	−
−	+	−
−	−	+

(5) Maxima & Minima



Point P, R T are points of local maxima
Point Q, S are point of local minima

Global maxima, global minima

Maximum = max $\{f(a), f(c_1), f(c_2) .... f(c_n), f(b)\}$

Minimum = min $\{f(a), f(c_1), f(c_2) .... f(c_n), f(b)\}$

$c_1\ c_2 .... c_n$ are n critical points.

(6) Test Of Local Maxima & Minima

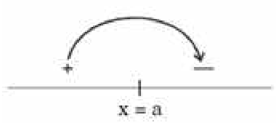
First Derivative Test:

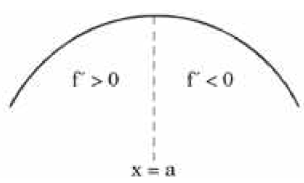
Step 1 : Find the critical points of the function by putting f'(x) = 0

Step 2 : For each of the critical points obtained in step 1 do the following :

Case 1 : x = a is local maxima

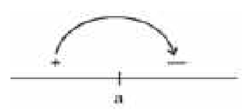
If f'(x) changes from + to - as x passes through a



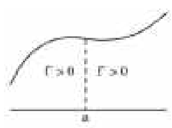


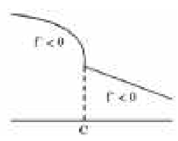
Case 2 : x = a is local minima

if the sign of f'(x) changes from - to + as x passes through a



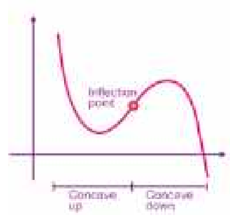
Case 3 : There is no sign change across a this means that x = a is neither a point of maxima nor minima.





(7) Point of inflection

A point of inflection is point where the curve changes its shape from convex to concave or from concave to convex.



(8) Higher Order Test

Let f be a differentiable function on interval I & let c be any point in the domain of f such that

(1) $f'(c) = f''(c) = f'''(c) \ldots = f^{n-1}(c) = 0$ and

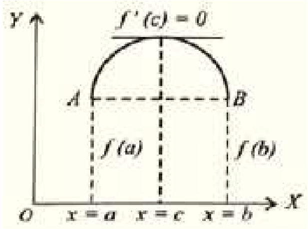
(2) $f^{n}(c) \neq 0$ and exists.

then if n is even $\begin{cases} f^{n}(c) < 0 \Rightarrow x = c \text{ is a local maxima} \\ f^{n}(c) > 0 \Rightarrow x = c \text{ is a local minima} \end{cases}$

(9) Rolle's Theorem

If a function f defined on the closed interval [a, b], is
1. continuous on [a, b],
2. derivable on (a, b) and
3. f (a) = f (b), then there exists atleast one real number c between a and b (a < c < b) such that f '(c) = 0.

Geometrical interpretation



Let the curve y = f (x), which is continuous on [a, b] and derivable on (a, b), be drawn.

The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.

Algebraic interpretation

Between two zeros a and b of f (x) (i.e., between two roots a and b of f (x) = 0) there exists atleast one zero of f '(x).

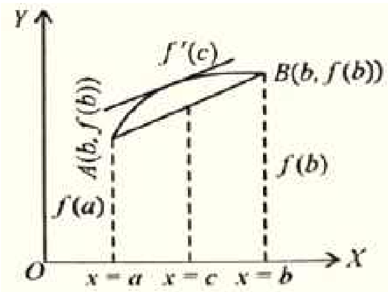
Key Points to Remember

1. The value of c may not be unique i.e., there can be more than one such c.
2. Every polyonomial function is continuous and differentiable for all real x.
3. The function log x is continuous on (0, ∞).
4. |x − a| is not differentiable at x = a (e.g., |x| is not differentiable at x = 0).
5. If the derivative of a function has finite and unique value on an interval, then the function is derivable on that interval.

(10) Lagrange's Mean Value Theorem

If a function f defined on the closed interval [a, b], is
1. continuous on [a, b] and
2. derivable on (a, b), then there exists atleast one real number c between a and b (a < c < b) such that

$$f(c) = \frac{f(b) - f(a)}{b - a}$$



Geometrical interpretation
The theorem states that between two points A and B on the graph off there exists atleast one point where the tangent is parallel to the chord AB.

(11) Parametric coordinates

(1) $x^{2/3} + y^{2/3} = a^{2/3}$: x = a $\cos^3\theta$, y = a $\sin^3\theta$.

(2) $\sqrt{x} + \sqrt{y} = \sqrt{a}$: x = a $\cos^4\theta$, y=a $\sin^4\theta$

(3) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$: x = a $(\cos\theta)^{2/n}$, y = b$(\sin\theta)^{2/n}$

(4) $c^2(x^2 + y^2) = x^2y^2$: x = c sec θ , y = c cosec θ

(5) $y^2 = x^3$: x = t^2 , y = t^3

(12) Angle of intersection of two Curves

The angle between the tangents to the two curves at their point of intersection.

Let c_1 & c_2 be two curves.

$$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{c_1}$$

$$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{c_2}$$

Angle of intersection, $\theta = \tan^{-1}\left|\dfrac{\left(\frac{dy}{dx}\right)_{c_1} - \left(\frac{dy}{dx}\right)_{c_1}}{1 + \left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2}}\right|$

(13) Orthogonal Curves

If the angle of intersection of two curves is a right angle, the two curves are said to be orthogonal. if the curves are orthogonal,

$$\left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2} = -1$$

(14) Subtangent & Subnormal

Length of Tangent $= \left| y_1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2_{x_1, y_1}} \right|$

Length of Normal $= \left| y_1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2_{x_1, y_1}} \right|$

Length of Subtangent $= \left| y_1 \left(\frac{dx}{dy}\right) x_1, y_1 \right|$

(Projection of tangent)

Length of subnormal $= \left| y_1 \left(\frac{dy}{dx}\right)_{x_1, y_1} \right|$

(Projection of normal)

(15) Leibnitz-rule

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)dt\right] = f(\Psi(x))\left\{\frac{d}{dx}\Psi(x)\right\} - f(\phi(x))\left\{\frac{d}{dx}\phi(x)\right\}$$

(16) Extrema of discontinuous Functions

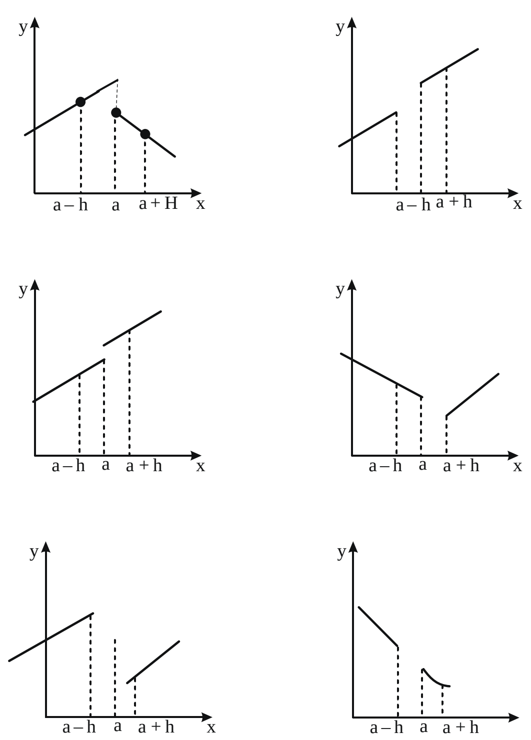
1. Minimum of discontinuous Functions :

For Minimum at x = a
f(a) ≤ f(a + h)
f(a) ≤ f(a − h)

2. Maximum of discontinuous Functions:

For maximum at x = a
f(a) ≥ f(a + h)
f(a) ≥ f(a − h)

3. Neither Maximum nor minimum exists:



Indefinite Integral

Let $f(x)$ be a function, the family of all its primitives (or antiderivatives) is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)dx$

1. Standard Intergrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ (ii) $\int \frac{1}{x} dx = \log|x| + C$ (iii) $\int e^x dx = e^x + C$ (iv) $\int a^x dx = \frac{a^x}{\log a} + C$ (v) $\int \sin x dx = -\cos x + C$ (vi) $\int \cos x dx = \sin x + C$

(vii) $\int \sec^2 x dx = \tan x + C$ (viii) $\int \operatorname{cosec}^2 x dx = -\cot x + C$ (ix) $\int \sec x \tan x dx = \sec x + C$ (x) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$ (xi) $\int \cot x dx = \log|\sin x| + C$

(xii) $\int \tan x dx = \log|\sec x| + C$ (xiv) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$ (xv) $\int \sec x dx = \log|\sec x + \tan x| + C$ (xvi) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvii) $\int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$ (xviii) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$ (xx) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$ (xix) $\int -\frac{1}{a^2 + x^2} dx = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + C$

(xxi) $\int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

2. Integration By Substitution

Expression	Substitution
$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or, $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

3. Integration Using Partial Fractions

(i) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$ (ii) $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further.

4. Integration By Parts

$\int u \cdot v dx = u\int v dx - \int\left[\frac{du}{dx} \cdot \int v dx\right]dx$

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- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$
- $\int\left[xf'(x) + f(x)\right]dx = xf(x) + C$
- $\int\sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int\sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int\sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

6. Integrals of different forms:

(1) $\int \sin^m x dx, \int \cos^m x dx$, where $m \leq 4$

express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using the following identities:

(i) $\sin^2 x = \frac{1-\cos 2x}{2}$ (ii) $\cos^2 x = \frac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ (iv) $\cos 3x = 4\cos^3 x - 3\cos x$

(2) $\int \sin mx \cos nx dx, \int \sin mx \sin nx dx, \int \cos mx \cos nx dx$

use the following trigonometrical identities:

$2\sin A\cos B = \sin(A+B) + \sin(A-B); 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B); 2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(3) $\int \frac{f'(x)}{f(x)}dx = \log\{f(x)\} + C$ (4) $\int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1}, n \neq -1$

(5) $\int \tan^m x\sec^{2n} x dx, \int \cot^m x\operatorname{cosec}^{2n} x dx; m, n \in N$

Put $\tan x = t$ and $\sec^2 x dx = dt$

(6) $\int \sin^m x\cos^n x dx, m, n \in N$

If the exponent of sin x is an odd positive integer put cos$x = t$

If the exponent of cos x is an odd positive integer put sin$x = t$.

(7) $\int \sin^m x \cos^n x\,dx$, Where $m,n \in Q$, $m+n$ is a negative even integer

Change the integrand in terms of $\tan x$ and $\sec^2 x$ by dividing numerator and denominator by $\cos^k x$, where $k=-(m+n)$ then put $\tan x = t$

(8) $\int \frac{1}{ax^2+bx+c}dx$

express ax^2+bx+c as the sum or difference of two squares.

(9) $\int \frac{px+q}{ax^2+bx+c}dx$

To evaluate this,

$$px+q=\lambda\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(10) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

In order to evaluate this, write

$$px+q=\lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(11) $\int \frac{P(x)}{ax^2+bx+c}dx$, where p (x) is a polynomial of degree two or more

to evaluate this, write $\int \frac{P(x)}{ax^2+bx+c}dx=\int Q(x)dx+\int \frac{R(x)}{ax^2+bx+c}dx$

(12) $\int \frac{1}{a\sin^2 x+b\cos^2 x}dx, \int \frac{1}{a+b\sin^2 x}dx$

$\int \frac{1}{a+b\cos^2 x}dx, \int \frac{1}{(a\sin x+b\cos x)^2}dx, \int \frac{1}{a+b\sin^2 x+c\cos^2 x}dx$

To evaluate this type of integrals, divide numerator and denominator both by $\cos^2 x$

(13) $\int \frac{1}{a\sin x+b\cos x}dx, \int \frac{1}{a+b\sin x}dx, \int \frac{1}{a+b\cos x}dx$

$\int \frac{1}{a\sin x+b\cos x+c}dx$

To evaluate this, put $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$ and, $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

and simplify.

(14) $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$

To evaluate this, write Numerator $=\lambda$(Diff. of denominator)) $+\mu$(Denominator))

(15) $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$

(16) $\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$

(17) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

In order to evaluate this, write

$$px+q=\lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu \text{ i.e. } px+q=\lambda(2ax+b)+\mu$$

(18) $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$

put $cx+d=t^2$

(19) $\int \frac{1}{\left(ax^2+bx+c\right)\sqrt{px+q}}dx$

put $px+q=t^2$

(20) $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}dx$

put $ax+b=\frac{1}{t}$

(21) $\int \frac{1}{\left(ax^2+b\right)\sqrt{cx^2+d}}dx$

put $x=\frac{1}{t}$ to obtain

(22) $\int \frac{-tdt}{\left(a+bt^2\right)\sqrt{c+dt^2}}$

substitute $c+dt^2=u^2$

7. Reduction formulas

(1) Reduction Formula for Exponential Functions

- $\int x^n e^{mx}dx=\left[(1/m)x^n e^{mx}\right]-\left[(n/m)\int x^{n-1}e^{mx}\right]dx$
- $\int e^{mx}/x^n dx=-\left[e^{mx}/(n-1)x^{n-1}\right]+\left[(m/n-1)\int e^{mx}/x^{n-1}\right]dx, n\neq 1$

(2) Reduction Formula for Trigonometric Functions

- $\int \sin^n(x)dx=\frac{-\mathrm{Sin}^{n-1}(x)\mathrm{Cos}(x)}{n}+\frac{n-1}{n}\int \mathrm{Sin}^{n-2}(x)dx$
- $\int x^n\cos(x)dx=x^n\sin(x)-n\int x^{n-1}\sin(x)dx$
- $\int x^n\sin(x)dx=-x^n\cos(x)+n\int x^{n-1}\cos(x)dx$
- $\int \sin^n(x)\cos^m(x)dx=\frac{\sin^{n+1}(x)\cos^{m-1}(x)}{n+m}+\frac{m-1}{n+m}\int \sin^n(x)\cos^{m-2}(x)dx$
- $\int \frac{dx}{\sin^n x}=-\frac{\cos x}{(n-1)\sin^{n-1}x}+\frac{(n-2)}{(n-1)}\int \frac{dx}{\sin^{n-2}x}, n\neq 1$
- $\int \frac{dx}{\cos^n x}=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{(n-2)}{(n-1)}\int \frac{dx}{\cos^{n-2}x}, n\neq 1$
- $\int \tan^n(x)dx=\frac{\tan^{n-1}(x)}{n-1}-\int \tan^{n-2}(x)dx$

(3) Reduction Formula for Logarithmic Functions

- $\int x^n \ln^m x\,dx=\frac{x^{n+1}\ln^m x}{n+1}-\frac{m}{n+1}\int x^n\ln^{m-1}x\,dx$
- $\int \frac{\ln^m x}{x^n}=-\frac{\ln^m x}{(n-1)x^{n+1}}+\frac{m}{n-1}\int \frac{l^{m-1}x}{x^n}dx, n\neq 1$

(4) Reduction Formula for Inverse Trigonometric Functions

- $\int x^n\arcsin x\,dx=\frac{x^{n+1}}{n+1}\arcsin x-\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$
- $\int x^n\arccos x\,dx=\frac{x^{n+1}}{n+1}\arccos x+\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$
- $\int x^n\arctan x\,dx=\frac{x^{n+1}}{n+1}\arctan x-\frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1+x^2}}dx$

(5) Reduction Formula for Algebraic Functions

- $\int \frac{dx}{\left(ax^2+bx+c\right)^n} = \frac{-2ax-b}{(n-1)\left(b^2-4ac\right)\left(ax^2+bx+c\right)^{n-1}} - \frac{2(2n-3)a}{(n-1)\left(b^2-4ac\right)}\int \frac{dx}{\left(ax^2+bx+c\right)^{n-1}}, n \neq 1$

- $\int \frac{dx}{\left(x^2+a^2\right)^n} = \frac{x}{2(n-1)a^2\left(x^2+a^2\right)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{\left(x^2+a^2\right)^{n-1}}, n \neq 1$

- $\int \frac{dx}{\left(x^2-a^2\right)^n} = \frac{x}{2(n-1)a^2\left(x^2a^2\right)^{n-1}} - \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{\left(x^2-a^2\right)^{n-1}}, n \neq 1$

8. Derived substitutions:

A. Algebraic Twins

- $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx,$
- $\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$
- $\int \frac{2x^2}{\left(x^4+1+kx^2\right)}dx \cdot \int \frac{2}{\left(x^4+1+kx^2\right)}dx$

B. Trigonometric twins

- $\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx,$
- $\int \frac{1}{\left(\sin^4 x+\cos^4 x\right)}dx, \int \frac{1}{\left(\sin^6 x+\cos^6 x\right)}dx, \int \frac{\pm\sin x \pm \cos x}{a+b\sin x\cos x}dx.$

Definite Integral

1. Definite Integral As The Limit Of A Sum

$$\int_a^b f(x)dx = \lim_{h\to 0}[f(a)+f(a+h)+----+f(a+(n-1)h)]$$

$$h=\frac{b-a}{n}\to 0 \text{ as } n\to\infty$$

The above expression is known as the definite integral as the limit of a sum.

2. Properties Of Definite Integrals

(i) $\int_a^b f(x)dx=\int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx=-\int_b^a f(x)dx$ in particular $\int_a^a f(x)dx=0$

(iii) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$ where $a<c<b$

(iv) $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

(vi) $\int_{-a}^a f(x)dx=2\int_0^a f(x)dx$ iff $f(x)$ is an even function

(vii) $\int_{-a}^a f(x)dx=0$, if $f(x)$ is an odd function

(viii) $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$

(ix) $\int_0^{2a} f(x)dx=2\int_0^a f(x)dx$ if $f(2a-x)=f(x)$
$=0$, if $f(2a-x)=-f(x)$

3. Walli's formula

- $\int_0^{\pi/2}\sin^n x dx=\int_0^{\pi/2}\cos^n x dx=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\ldots\frac{2}{3}, & \text{when } n \text{ is odd}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\ldots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & \text{when } n \text{ is even}\end{cases}$

- $\int_0^{\pi/2}\sin^m x\cos^n x dx=\frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots(2 \text{ or } 1)}$

[If m, n are both odd positive integers or one odd positive integer]

- $\int_0^{\pi/2}\sin^m x\cos^n x dx=\frac{(m-1)(m-3)\ldots\ldots\ldots(n-1)(n-3)}{(m+n)(m+n-2)\ldots\ldots(2 \text{ or } 1)}\cdot\frac{\pi}{2}$

[If m, n are both even positive integers]

4. Periodic Properties

If f(x) is a periodic function with period T, then

01 $\int_0^{nT} f(x)dx=n\int_0^T f(x)dx, n\in Z$

02 $\int_a^{a+nT} f(x)dx=n\int_0^T f(x)dx, n\in Z, a\in R$

03 $\int_{mT}^{nT} f(x)dx=(n-m)\int_0^T f(x)dx, m,n\in Z$

04 $\int_{nT}^{a+nT} f(x)dx=\int_0^a f(x)dx, n\in Z, a\in R$

5. Advance properties

$\psi(x)\le f(x)\le\phi(x)$ for $a\le x\le b$ then $\int_a^b\psi(x)dx\le\int_a^b f(x)dx\le\int_a^b\phi(x)$

01 If $m\le f(x)\le M$ for $a\le x\le b$, then $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$

02 $\left|\int_a^b f(x)dx\right|\le\int_a^b|f(x)|dx$

03 If $f(x)\ge 0$ on $[a,b]$ then $\int_a^b f(x)dx\ge 0$

6. Leibnitz Theorem

if $F(x)=\int_{g(x)}^{h(x)} f(t)dt$, then

$$\frac{dF(x)}{dx}=h'(x)f(h(x))-g'(x)f(g(x))$$

7. Beta & Gama Function

Gamma function

$\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$

where n is a positive rational number

Properties of gamma function

1) $\Gamma(0)=\infty, \Gamma(1)=1$

2) $\Gamma(n+1)=n\Gamma(n)$

3) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

4) $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}, 0<n<1$

Beta function

$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

NOTE The relationship between beta & gamma function will be $\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$

8. Important results

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{n=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

In GP, sum of n terms, $S = s_n = \begin{cases} \frac{a(r^n-1)}{r-1}, & |r|>1 \\ an, & r=1 \\ \frac{a(1-r^n)}{1-r}, & |r|<1 \end{cases}$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}, \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$\sin\alpha + \sin(\alpha+1\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+\overline{(n-1)}\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \cdot \sin(\alpha+(n-1)\beta/2).$$

$$\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+\overline{(n-1)}\beta) = \frac{\sin n\beta/2}{\sin\beta/2} \cdot \cos(\alpha+(n-1)\beta/2)$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots\infty = \frac{\pi^2}{12}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\infty = \frac{\pi^2}{6}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\quad\infty = \frac{\pi^2}{24}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$1/1^2 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots\infty = \pi^2/8$$

9 Average Value Theorem

If f is a continuous function on [a, b], then its average value on [a, b] is given by the formula

$$f_{\text{AVG}[a,b]} = \frac{1}{b-a} \cdot \int_a^b f(x)dx$$

Application Of Integrals

1

$\int_a^b f(x)dx \neq$ Area under the curve f(x) from a to b

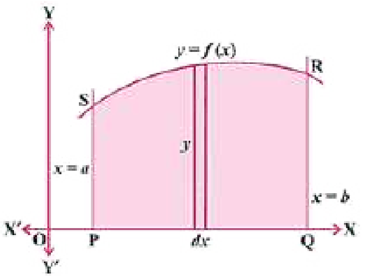
$\int_a^b f(x)dx =$ Algebraic area under the curve f(x) from a to b

2 POSITIVE AND NEGATIVE AREA

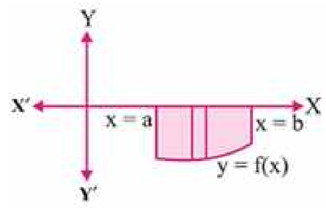
Area is always taken as positive. If some part of the area lies in the positive side i.e., above x-axis and some part lies in the negative side i.e. below x-axis then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

3 Area Under Simple Curves

(i) Area of the region bounded by a curve y = f(x) and x-aixs between the two ordinates



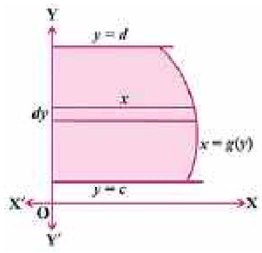
Area, $A = \int_a^b dA = \int_a^b y dx = \int_a^b f(x) dx$



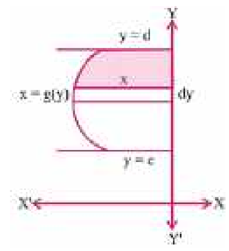
If the position of the curve under consideration is below the x -axis. Then, area is negative. So, we take its absolute value, i.e.,

$\text{Area}(A) = \left|\int_a^b f(x) dx\right|$

(ii) Area of the region bounded by a curve x = f(y) and x-aixs between the two ordinates



Area, $A = \int_c^d x dy = \int_c^d g(y) dy$

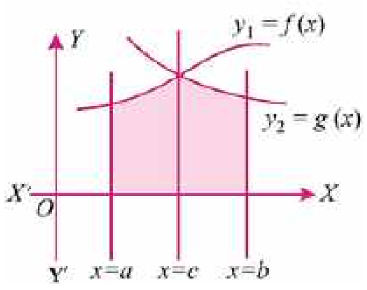


If the position of the curve under consideration is below the y-axis. Then, area is negative. So, we take its absolute value, i.e.,

$\text{Area}(A) = \left|\int_c^d g(y) dy\right|$

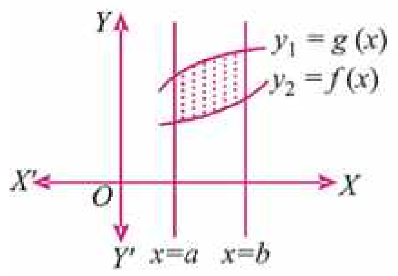
4 Area Under Different Curves

CASE-I



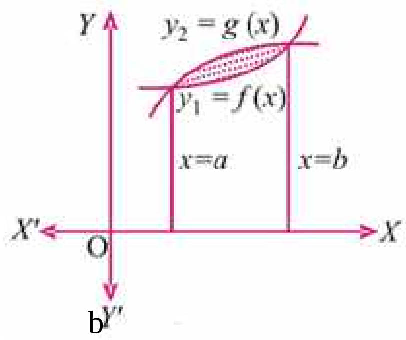
$$A = \int_a^c f(x)dx + \int_c^b g(x)dx$$

CASE-II



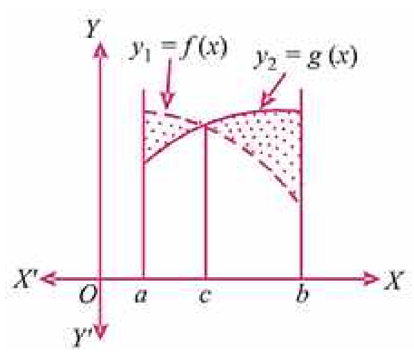
$$A = \int [g(x) - f(x)]dx$$

CASE-III



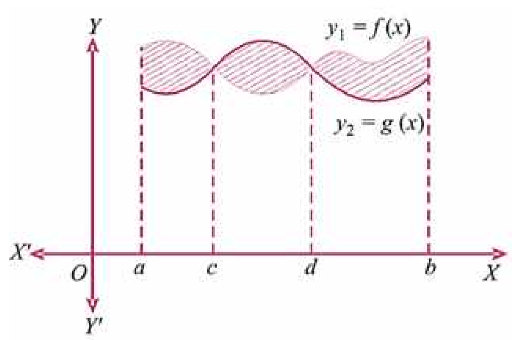
$$A = \int_a^b [g(x) - f(x)]dx$$

CASE-IV



$$A = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

CASE-V



$$A = \int_a^c (y_1 - y_2)dx + \int_c^d (y_2 - y_1)dx + \int_d^b (y_1 - y_2)dx$$

DIFFERENTIAL EQUATION

1 Order Of Differential Equation

The order of a differential equation is the order of the highest derivative occuring in the differential equation.
For example

$\frac{d^2y}{dx^2}+y=0$ is a second order differential equation

$\left(\frac{d^3y}{dx^3}\right)+x^2\left(\frac{d^2y}{dx^2}\right)^3=0$ is a third order differential equation.

2 Degree Of Differential Equation

The degree of a differential equation is the highest degree of the highest derivative occuring in the differential equation when it is a polynomial of the differential coefficients i.e., differential coefficients free from radicals & fractions.

For example

Since, $\frac{d^3y}{dx^3}+x^2\left(\frac{d^2y}{dx^2}\right)^3=0$ as order = 3

its degree = 1, as $\frac{d^3y}{dx^3}$ has power 1.

3 Differential Equations With Variables Separable

$\frac{dy}{dx}=h(y)\cdot g(x)$

Separating the variables, we have $\frac{dy}{h(y)}=g(x)\cdot dx$

Integrate both sides $\int\frac{dy}{h(y)}=\int g(x)\cdot dx$

4 Reducible to the separate variable type

$\frac{dy}{dx}=f(ax+by+c)$ is solved by putting ax + by + c = t, etc

5 Homogenous differential equation

(i) P(x, y)dx + Q(x, y)dy = 0 is called homogenous, if P & Q are homogenous functions of the same degree on x & y. Reducible to $y'=f\left(\frac{y}{x}\right)$

substitute y = xu, u is unknown function. The equation is transformed to an equation with variable separables.

(ii) $\frac{dy}{dx}=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, $a_1b_2-a_2b_1\neq 0$, then substitute x = u + h, y = v + k

if $a_1b_2-a_2b_1=0$, $u=a_1x+b_1y$ transforms into a variable separable form.

- P(x, y) function is homogenous of degree n, if for any real t, $P(tx, ty) = t^n(P(x, y))$.
- A differential equation of the form $\frac{dy}{dx}=f(x,y)$ is homogeneous, if f(x, y) is a homogeneous function of degree zero i.e., $f(tx, ty) = t^0.f(x,y)$

6 Exact differential equation

M(x, y)dx + N(x, y)dy = 0 is exact if its LHS expression is the exact differential of some function u(x, y).

du = Mdx + Ndy

Then solution is u(x, y) = c.

The sufficient condition for the differential Mdx + Ndy = 0 to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

The solution of Mdx + Ndy = 0 is

$\int_{y-\text{constant}} Mdx+\int(\text{terms of N not containing x})dy=c$, provided

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

7 Linear Differential Equations

A differential equation of the form $\frac{dy}{dx}+Py=Q$

where P & Q are constants or functions of only, is known as a First Order Linear Differential Equation.

$y(\text{I.F})=\int(Q\times \text{I.F})dx+c$

$\frac{dx}{dy}+P'x=Q'$

$\text{I.F}=e^{\int Pdy}$

$x(\text{I.F})=\int\left(Q'\times \text{I.F}\right)dy+c$

8 After linear differential equation

Solution by inspection

(1) d(xy) = xdy + ydx

(2) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(3) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(4) $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$

(5) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(6) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(7) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(8) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-2dy}{x^2+y^2}=\frac{d(x/y)}{1+(x/y)^2}$

(9) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}=\frac{d(y/x)}{1+(y/x)^2}$

(10) $d[\ln(xy)]=\frac{xdy+ydx}{xy}$

(11) $d\left[\ln\left(\frac{x}{y}\right)\right]=\frac{ydx-xdy}{xy}$

(12) $d[\ln(y/x)]=\frac{xdy-ydx}{xy}$

(13) $d\left[1/2\ln\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$

(14) $d(-1/xy)=\frac{xdy+ydx}{x^2y^2}$

(15) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(16) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

(17) $\frac{d[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{(f(x,y))^n}$

9 Bernoulli's equation

$\frac{dy}{dx}+Py=Qy^n$ dividing $y^n\rightarrow y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ ...(i)

$y^{1-n}=z$

(i) $\frac{dz}{dx}+(1-n)Pz=(1-n)Q$

Solution is

$ze^{\int(1-n)Pdx}=\int\left\{(1-n)Q\cdot e^{\int(1-n)Pdx}\right\}dx$

10 Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angles, is called an orthogonal trajectory of the family.

Procedure for finding Orthogonal Trajectory :

(i) Let f(x, y, c) = 0 is the equation of family.

(ii) Differentiate f = 0, w.r.t. x & eliminate c.

(iii) Substitute $-\frac{dx}{dx}$ for $\frac{dy}{dx}$ That is the differential equation of OT. Now, solve it to get OT.

11 Clairaut's equation

Form y = px + f(p)

Method. Differentiate w.r.t. x, we get

$$\{x + f'(p)\}\frac{dp}{dx} = 0$$

$\therefore$ **p = c or f '(p) + x = 0**

When p = c, the general solution is y = cx + f(c) which gives a family of straight lines

When f '(p) + x = 0, eliminating p from y = px + f(p) and f '(p) + x = 0 we get a solution which is a curve (without any arbitrary constant) touching all the lines given by y = cx + f(c). This solution is called the singular solution.

12 Facts from cartesian curve

(i) Slope of tangent at any point $P(x, y) = \frac{dy}{dx}$

(ii) Equation of tangent PQ at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$

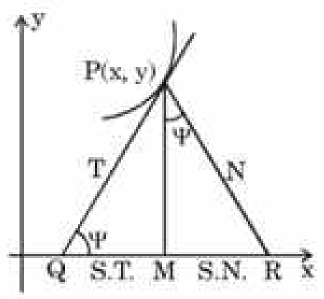
(iii) Equation of normal PR at (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$

(iv) Length of tangent PQ at $(x, y) = \left|y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right|$

(v) Length of normal PG at $(x, y) = \left|y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|$

(vi) Length of subtangent QM at $(x, y) = \left|y \cdot \frac{dx}{dy}\right|$

(vii) Length of subnormal MR at $(x, y) = \left|y \cdot \frac{dy}{dx}\right|$



VECTOR ALGEBRA

01 Types Of Vectors

1. Zero Vector : A vector whose initial and terminal points coincide, It has zero magnitude.
2. Unit Vector: A vector whose magnitude is unity. The unit vector in the direction of $\vec{a}$ is denoted as $\hat{a}$.
3. Coinitial Vectors : Two or more vectors having the same initial point.
4. Collinear Vectors: Two or more vectors are collinear, if they are parallel to the same line irrespective of their magnitude.
5. Equal Vectors: Two vectors are said to be equal, if they have same magnitude & direction regardless of the position of their initial points.
6. Negative of a vector : A vector whose magnitude is the same as that of the given vector, but the direction is opposite to that of it.
7. Position Vector: Let O be the origin & P(X,Y,Z) be a point with respect to the origin O. Then the vector called the position vector of the point P with respect to O. $\overline{OP}$ is

$$|\overline{OP}| = \sqrt{x^2+y^2+z^2}$$

- Direction angles: The angles made by $\overline{OP}$ with positive direction of x, y, & z-axes (say α, β & γ respectively).
- Directions cosines: the cosine value of these angles i.e., $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ of $\overline{OP}$ denoted by l, m & n respectively.

02 Properties of Vector Addition

(i) For any two vectors $\vec{a}$ & $\vec{b}$, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative property)
(ii) For any three vectors $\vec{a},\vec{b}$, & $\vec{c}$ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (Associative property)

03 Multiplication Of A Vector By A Scalar

if $\vec{a}$ is multiplied by scalar m then the product $m\vec{a}$ is a vector whose magnitude is $|m|$ times that of $\vec{a}d$ & direction is same as $\vec{a}$ if m is positive where as opposite to that of $\vec{a}$ if m is negative.

- $m(\vec{a})=(\vec{a})m$
- $(m+n)\vec{a}=m\vec{a}+n\vec{a}$
- $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$
- $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}.$

04 Dot or Scalar Product of Vectors

Dot product of two vectors $\vec{a}$ & $\vec{b}$ inclined at an angle θ is $(\vec{a}\cdot\vec{b})=|\vec{a}||\vec{b}|\cos\theta$

- $\vec{a}\cdot\vec{b}\in R$
- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- $(x\vec{a})\cdot\vec{b}=x(\vec{a}\cdot\vec{b})=\vec{a}\cdot(x\vec{b})$
- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
- If $\vec{a}$ & $\vec{b}$ perpendicular, $\vec{a}\cdot\vec{b}=0$
- $\vec{a}\cdot\vec{b}<0$ iff angle between $\vec{a}$ & $\vec{b}$ is obtuse.
- $\hat{i}\cdot\hat{i}=1$, $\hat{j}\cdot\hat{j}=1$, $\hat{k}\cdot\hat{k}=1$, $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{i}\cdot\hat{k}=0$
- If two vectors have same direction then $\cos\theta=1\Rightarrow\vec{a}\cdot\vec{b}=ab$
- If two vectors have opposite direction then $\cos\theta=-1\Rightarrow\vec{a}\cdot\vec{b}=-ab$
- If $\hat{a}$ & $\hat{b}$ are unit vectors, $\hat{a}\cdot\hat{b}=\cos\theta$
- $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ $\vec{a}\cdot\vec{b}=a_1b_1+b_1b_2+c_1b_3.$
- Projection of a vector $\vec{b}$ on the other vector $\vec{a}$ is given by $\vec{b}\cdot\hat{a}$ or $\vec{b}\left(\frac{\vec{a}}{|\vec{a}|}\right)$
- A vector in the direction of the bisector of the angle between the two rectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}$
- Bisector of the interior angle between two vectors $\vec{a}$ & $\vec{b}$ is $\lambda\left(\frac{\vec{a}}{a}+\frac{\vec{b}}{b}\right)$ i.e., $\lambda(\hat{a}+\hat{b})$ where $\lambda\in R^+$ & Bisector of the interior angle is $\lambda\left(\frac{\vec{a}}{a}-\frac{\vec{b}}{b}\right)$, is $\lambda(\hat{a}-\hat{b})$

05 Cross product

Let $\vec{a}$ & $\vec{b}$ be two non-zero vectors inclined at an angle θ
Then, vector product is defined as $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$
where, $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ & $\vec{b}$ such that $\vec{a},\vec{b}$ & $\hat{n}$ form a right handed system.

- **Lagrange's Identity:**
  For any two vectors $\vec{a},\vec{b}$

$$(\vec{a}\times\vec{b})^2=a^2b^2-(\vec{a}\cdot\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$$

- **Formulation of vector product in terms of scalar product:** The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

$$|\vec{c}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$$

$\vec{c}\cdot\vec{a}=\vec{c}\cdot\vec{b}=0$
$\vec{a},\vec{b},\vec{c}$ form a right-handed system.

- **Remarks**

(a) $\vec{a}\times\vec{b}$ is a vector.
(b) If $\vec{a}$ & $\vec{b}$ are nonzero vectors, then $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}\parallel\vec{b}$
(c) For mutually perpendicular unit vectors $\hat{i},\hat{j},\hat{k}$,
$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$
(d) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
(e) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a triangle then its area is $\frac{1}{2}|\vec{a}\times\vec{b}|$
(f) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a parallelogram then the area is $|\vec{a}\times\vec{b}|$
(g) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$
(h) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
(i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n}=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

- **Vector area:**
- If $\vec{a},\vec{b}$ & $\vec{c}$ are the position vectors of 3 points then area of △ABC $=\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$ A, B, C are collinear iff $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0.$
- Area of any quadrilateral whose diagonal vectors are d_1 & d_2 is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$

06 Vector Triple Product:

Vector Triple Product of $\vec{a},\vec{b},\vec{c}$ is $\vec{a}\times(\vec{b}\times\vec{c})$.
It is a vector perpendicular to the plane containing $\vec{a}$ & $\vec{b}\times\vec{c}$ lying in the plane of $\vec{b}$ & $\vec{c}$

- $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$
- $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

07 Test of Collinearity

$x\vec{a}+y\vec{b}+z\vec{c}=0$ [x, y, z scalars, x + y + z = 0]

08 Test of Coplanarity

$x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$ [x, y, z, w scalars, x + y + z + w = 0]

09 Reciprocal system of Vectors

If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of noncoplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$ then the two systems are called reciprocal systems.

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

10 Scalar Triple Product/Box Product: $[\vec{a}\ \vec{b}\ \vec{c}]$

Box product of $\vec{a},\vec{b},\vec{c}$ is $(\vec{a}\times\vec{b})\cdot\vec{c}=abc\sin\theta\cos\phi$

$\theta\rightarrow$ angle between $\vec{a}$ & $\vec{b}$

$\phi\rightarrow$ angle between $\vec{a}\times\vec{b}$ and $\vec{c}$

Box product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a},\vec{b},\vec{c}$

$$V=[\vec{a}\ \vec{b}\ \vec{c}]$$

- $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$
- $[\vec{a}\vec{b}\vec{c}]=[\vec{b}\vec{c}\vec{a}]=[\vec{c}\vec{a}\vec{b}]$.
- $\vec{a}\cdot(\vec{b}\times\vec{c})=-\vec{a}\cdot(\vec{c}\times\vec{b})$.
- $[\vec{a}\vec{b}\vec{c}]=-[\vec{a}\vec{c}\vec{b}]$.
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

$\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

- $\vec{a},\vec{b},\vec{c}$ are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}]=0$
- If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}]>0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}]<0$ for left handed system.
- $[\hat{i}\ \hat{j}\ \hat{k}]=1$
- $[k\vec{a}\ \ \vec{b}\ \ \vec{c}]=k[\vec{a}\ \ \vec{b}\ \ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \ \vec{c}\ \ \vec{d}]=[\vec{a}\ \ \vec{c}\ \ \vec{d}]+[\vec{b}\ \ \vec{c}\ \ \vec{d}]$

11 Direction cosines & Direction Ratios

- If $\vec{a}$ makes angles of α,β,γ with the direction of x, y, z axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines which $\vec{a}$ is are usually denoted by l, m, n.
- Any three members a,b,c proportional to the direction cosines of a line are called direction ratios

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

$$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}};\quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}},\ l^2+m^2+n^2=1$$

12 Vector Equation of a Line

- Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$
- If line passes through the point $A(\vec{a})$ & is parallel to $\vec{b}$, then its equation is $\vec{\gamma}=\vec{a}+t\vec{b}$
- Equation of the bisectors of the angle between the lines,

$\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}=\vec{a}+\mu\vec{c}$ is $\vec{r}'=\vec{a}+t(\vec{b}+\vec{c})$ & $\vec{r}=\vec{a}+p(\vec{c}-\vec{b})$

13 Shortest distance between two lines

If two lines are $\vec{r}_1=\vec{a}_1+k\vec{b}$ & $\vec{r}_2=\vec{a}_2+k\vec{b}$

then $\alpha=\left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

14 Equation of Plane

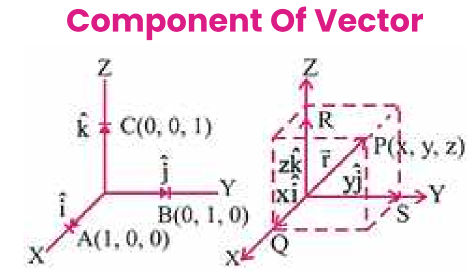
$(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$ containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.

$\vec{r}\cdot\vec{n}=d$ general equation

15 Projection & Component

- Projection of $\vec{a}$ along $\vec{b}=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
- Component of $\vec{a}$ along $\vec{b}=\dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$
- Projection of $\vec{a}\perp\vec{b}=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{b}|}$
- Component of $\vec{a}\perp\vec{b}=\vec{a}-\dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

16 Component Of Vector



$\overline{OA},\overline{OB}$ & $\overline{OC}$ are unit vectors along x,y & z axes respectively, denoted by $\hat{i},\hat{j}$ & $\hat{k}$ respectively Position Vector of with reference to O is given by:

$$\overline{OP}(\text{ or }\vec{r})=x\hat{i}+y\hat{j}+z\hat{k}.$$

This form of any vector is called its component form.

Also, $(\overline{OP})=|\vec{r}|=\sqrt{x^2+y^2+z^2}$

17 Vector Joining Two Points

Let $A(x_1,y_1,z_1)$ & $B(x_2,y_2,z_2)$ be any two points in the space, then

$$\overline{OA}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\ \&\ \overline{OB}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

$$\therefore\overline{AB}=\overline{OB}-\overline{OA}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

$$|\overline{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

18 Section Formulae

The position vector of a point R dividing a line segment joining the points points P & Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio m : n

(i) internally, is given by $\dfrac{m\vec{b}+n\vec{a}}{m+n}$

(ii) externally, is given by $\dfrac{m\vec{b}-n\vec{a}}{m-n}$

The position vector of the middle point of PQ is given by $\frac{1}{2}(\vec{a}+\vec{b})$

19 Scalar product of four vectors

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{b}.\vec{c}\\ \vec{a}.\vec{d} & \vec{b}.\vec{d}\end{vmatrix}$$

20 Vector product of four vectors

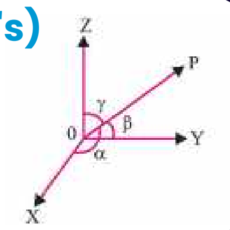
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

Three Dimensional Geometry

1. Direction Cosines of A Line (Dc's)

The direction cosines are generally denoted by l, m, n.

Hence, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$
Note that $l^2 + m^2 + n^2 = 1$



2. Direction Ratio of A Line (Dr's)

- Any three numbers a, b and c proportional to the direction cosines l, m and n, respectively are called direction ratios of the line.
- The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

3. Equation of Line

1. Equation of a line through a given point with position vector $\overline{a}$ and parallel to a given vector $\overline{b}$:

In vector form, $\overline{r} = \overline{a} + \lambda\overline{b}$

In cartesian form, $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ Here, a, b, c are also the direction ratios of the line.

2. Equation of a line passing through two given points with position vectors $\overline{a}$ and $\overline{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

In cartesian form, $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

4. Angle Between Two Lines

In vector form, The angle between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

In cartesian form, The angle between two lines :

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ is } \cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If two lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two lines are parallel, then $\vec{b}_1 = \lambda\vec{b}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

5. Shortest Distance Between Two Lines

1. Distance Between Parallel Lines the shortest distance between parallel lines

$L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

2. Distance Between Two Skew Lines In vector form, The distance between two skew lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

In cartesian form,
The distance between two skew lines :

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is:

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}$$

6. Equation of A Plane In Normal Form

Vector Form

$\vec{r} \cdot \hat{n} = d$

Here $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\hat{n}$ is the unit vector along the normal from origin to the plane.
d is perpendicular distance of the plane from the origin.

Cartesian Form

$lx + my + nz = d$

where ℓ, m, n are the direction cosines of $\hat{n}$ (unit vector along the normal from origin to the plane).

7. Equation Of A Plane Perpendicular To A Given Vector And Passing Through A Given Point

Vector Form

Let a plane pass through a point with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$. Then its equation is given as: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

Cartesian Form

Let a plane pass through a point $(x_1, y_1 z_1)$ & the direction ratio of the vector perpendicular to the plane be A, B, C. Then its equation is given as:

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

8. Equation Of A Plane Passing Through Three Non-Collinear Points

Vector Form

$[\vec{r}\vec{b}\vec{c}] + [\vec{r}\vec{a}\vec{b}] + [\vec{r}\vec{c}\vec{a}] = [\vec{a}\vec{b}\vec{c}]$ or $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

where, $\vec{a}, \vec{b}, \vec{c}$ are the position vector of three given noncollinear points through which the plane passes.

Cartesian Form

The equation of plane passing through three noncollinear points Y with coordinates, (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3) is given as:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

9. Intercept Form of The Equation of A Plane

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Where a, b, c are the intercepts made by the plane on x, y & z axes respectively.

10. Plane Passing Through The Intersection Of Two Given Planes

Vector Form

Equation of plane passing through the point of intersection of two planes

$\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given as:

$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1+\lambda d_2$

Cartesian Form

$\vec{n}_1 = A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$

$\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$ and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

therefore its cartesian equation is:

$(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0$

12. Angle Between Two Planes

Vector Form: The angle between two planes $\vec{r}\cdot\vec{n} = d_1$ & $\vec{r}\cdot\vec{n} = d_2$ is given as:

$$\cos\theta = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$

Cartesian Form: The angle between two planes

$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given as

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If two planes are perpendicular, then $\vec{n}_1\cdot\vec{n}_2=0$ or $a_1a_2+b_1b_2+c_1c_2=0$
- If two planes are perpendicular, then $\vec{n}_1=\lambda\vec{n}_2$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

11. Coplanarity of Two Lines

Vector Form

Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar, if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

Cartesian Form

Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

13. Distance Of A Point From A Plane

Vector Form

Distance of a point with position vector $\vec{a}$ from a plane $\vec{r}\cdot\vec{n}=d$ is given as: $\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$

Cartesian Form

Distance of a point (x_1, y_1, z_1) from a plane: $ax+by+cz=d$ is given as :

$$\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

14. Angle Between A Line And A Plane

Vector Form

Angle between a line

$\vec{r}=\vec{a}+\lambda\vec{b}$ and a plane $\vec{r}\cdot\vec{n}=d$ is

$$\sin\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Cartesian Form

Angle between a line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and a plane

$a_2x+b_2y+c_2z=d$ is given as:

$$\sin\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If line is perpendicular to the plane, then $\vec{n}=\lambda\vec{b}$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
- If line is parallel to the plane, then $\vec{n}\cdot\vec{b}=0$ or $a_1a_2+b_1b_2+c_1c_2=0$

A REASONABLE PROBABILITY IS THE ONLY CERTAINTY

-E.W.HOWE

PROBABILITY

1 Probability

If an experiment results in a total of (m + n) outcomes which are equally likely and if 'm' outcomes are favourable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A' denoted by P(A), is defined by

$$P(A) = \frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

2 Random Experiment

An Experiment is called random experiment if it satisfies the following two conditions:

1. It has more than one possible outcome.
2. It is not possible to predict the outcome in advance.

Outcome: A possible result of a random experiment is called its outcome.
Sample Space: Set of all possible outcomes of a random experiment is called sample space. It is denoted by symbol ' S '.

3 Algebra of Events

- Event A or B or $A \cup B = \{w : w \in A \text{ or } w \in B\}$
- Event A and B or $A \cap B = \{w : w \in A \text{ and } w \in B\}$
- Event A but not B or $A - B = A \cap B'$

4 Types of Events

1. Impossible and Sure Event: The empty set φ is called an Impossible event, where as the whole sample space ' S ' is called 'Sure event'.

2. Simple Event: If an event has only one sample point of a sample space, it is called a 'simple event'.

3. Compound Event: If an event has more than one sample point, it is called a 'compound event'.

4. Complementary Event: Complement event to A = 'not A'

5. Exhaustive Events: Many events that together form sample space are called exhaustive events.

6. Mutually Exclusive: Events A & B are called mutually exclusive events if occurrence of any one of them excludes occurrence of other event, i.e. they cannot occur simultaneously.

7. Mutually exclusive and exhaustive: The events which are not mutually exclusive are known as compatible events or mutually non exclusive events. Mutually exclusive and exhaustive system of events: Let S be the sample space associated with a random experiment. Let $A_1, A_2, ..., An$ be subsets of S such that
(i) $A_i \cap A_j = \varphi$ for $i \neq j$ and
(ii) $A_1 \cup A_2 \cup ... \cup A_n = S$
Then, the collection of events $A_1, A_2, ..., An$ is said to form a mutually exclusive and exhaustive system of events.

8. Independent Events

(i) If E&F are independent, then

$(P(E \cap F) = P(F)P(E/F) = P(E), P(F) \neq 0 P(F/E) = P(F), P(E) \neq 0)$

(ii) Three events A,B&C are said to be mutually independent, if

$(P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C), P(B \cap C) = P(B)P(C)$ & $P(A \cap B \cap C) = P(A)P(B)P(C))$

If at least one of the above is not true for three given events, we say that the events are not independent.

5 Probability of AUB, A ∩ B and P (notA)

If A and B are any two events, then

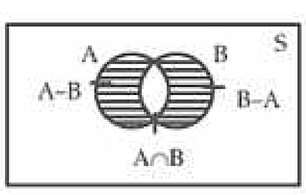
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(AUB)$$

If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

Probability of the event 'not A'

$$P(A') = P(\text{not } A) = 1 - P(A)$$



6 Conditional Probability

If E and F are two events associated with the sample space of a random experiment, the conditional probability of the event given that it has occured is given as:

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)}, P(F) \neq 0$$

Properties of Conditional Probability

1. Let E & F be events of sample space of an experiment, then we have
$$P(S/F) = P(F/F) = 1$$
2. If A and B are any two events of a sample space S & F is an event of such that
$P(F) \neq 0$, then $P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$
In particular if A and B are disjoint events, then
$$P((A \cup B)/F) = P(A/F) + P(B/F)$$
3. $P(E'/F) = 1 - P(E/F)$

7 Multiplication Theorem On Probability

For two events E & F associated with a sample space S, we have
$(P(E \cap F) = P(E)\ P(F/E) = P(F)P(E/F))$
provided $P(E) \neq 0$ & $P(F) \neq 0$
The above result is known as Multiplication Rule of Probability.

8 Total Probability Theorem

If an even A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, ..., Bn$ and the probabilities $P(A/B_1), P(A/B_2)... P(A/B_n)$ are known, then

$$P(A) = \sum_{i=1}^{n} P(B_i) P(\mathbf{A} | B_i)$$

9 Baye's Theorem

Partition of a Sample Space

A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi, i \neq j, i, j = 1, 2, 3, \ldots n$
(b) $E_1 \cup E_2 \cup \ldots \cup E_n = S$
(c) $P(E_i) > 0$ for all $i = 1, 2, \ldots n$

Theorem of Total Probability. Let $\{E_1, E_2, \ldots .. E_n\}$ be a partition of the sample space S and suppose that each of the events $E_1, E_2, \ldots .., E_n$ has nonzero probability of occurence. Let A be any event associated with S then

$$P(A) = \sum_{j=1}^{n} P(E_j) P(A/E_j)$$

Baye's Theorem: If $E_1, E_2, ..., E_n$ are non-empty events which constitute a partition of sample space S & A is any event of non-zero probability.

$$P(E_i/A) = \frac{P(E_i) P(A/E_i)}{\sum_{j=1}^{n} P(E_j) P(A/E_j)} \quad \text{for any } i = 1, 2, 3, \ldots, n$$

10 Random Variable & Its Probability Distributions

A random variable is a real valued function whose domain is the sample space of a random experiment. The probability distribution of a random variable

$$X: x_1 \quad x_2 \quad \cdots \quad x_n$$
$$P(X): p_1 \quad p_2 \quad \cdots \quad p_n$$

where, $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2, \ldots, n$

The real numbers $x_1, x_2, \ldots, x_n$ are the possible values of the random variable X and $p_i (i = 1, 2, \ldots, n)$ is the probability of the random variable i.e.,

$$P(X = x_i) = p_i$$

11 Mean Of A Random Variable

The mean (μ) of a random variable X is also called the expectation of X denoted by $E(X)$

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

Here $x_1, x_2, \ldots, x_n$ are possible values of random variable X, occuring with probabilities $p_1, p_2, \ldots, p_n$ respectively.

12 Variance Of Random Variable

Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p(x_1), p(x_2) \ldots, p(x_n)$ respectively. Also let $\mu = E(X)$ be the mean of X then the variance of X is given as:

$$\text{Var}(X) \text{ or } \sigma_X^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$$

The non-negative number $\sigma_X = \sqrt{\text{Var}(X)}$ is called the Standard Deviation of random variable X

13 Bernoulli Trials & Binomial Distribution

Bernoulli Trials :
Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:
(i) There should be a finite number of trials.
(ii) The trials should be independent.
(iii) Each trial has exactly two outcomes: success or failure.
(iv) The probability of success remains same in each trial.

Binomial Distribution :
The probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion $(q + p)^n$ where p is probability of success in each trial and p + q = 1 . Hence, this distribution (also called Binomial distribution B(n, p) of number of successes X can be written as:

X	0	1	2	---	x	n
P(x)	${}^nC_0 q^n$	${}^nC_1 q^{n-1} p^1$	${}^nC_2 q^{n-2} p^2$		${}^nC_x q^{n-x} p^x$	${}^nC_n p^n$

The probability of X successes $P(X = x)$ is also denoted by $P(x)$ is given as:

$$P(x) = {}^nC_x q^{n-x} p^x, x = 0, 1, \cdots, n \quad (q = 1 - p)$$

This $P(x)$ is called the probability function of the binomial distribution.