1 RELATIONS AND FUNCTIONS

KEY CONCEPT INVOLVED

1. **Relations** - Let A and B be two non-empty sets then every subset of A × B defines a relation from A to B and every relation from A to B is a subset of A × B.
   Let R ⊆ A × B and (a, b) ∈ R. then we say that a is related to b by the relation R as aRb. If (a, b) ∉ R as a ʀ̸ b.

2. **Domain and Range of a Relation** - Let R be a relation from A to B, that is, let R ⊂ A × B. then *Domain* R = {a : a ∈ A, (a, b) ∈ R for some b ∈ B} i.e. dom. R is the set of all the first elements of the ordered pairs which belong to R. *Range R* = (b : b ∈ B, (a, b) ∈ R for some a ∈ A} i.e. range R is the set of all the second elements of the ordered pairs which belong to R. Thus Dom. R ⊂ A, Range R ⊂ B.

3. **Inverse Relation** - Let R ⊂ A × B be a relation from A to B. Then inverse relation R^{-1} ⊂ B × A is defined by R^{-1} {(b, a) : (a, b) ∈R}
   It is clear that
   (i) $aRb = bR^{-1} a$
   (ii) dom. R^{-1} = range R and range R^{-1} = dom R.
   (iii) $(R^{-1})^{-1} = R$.

4. **Composition of Relation** - Let R ⊂ A × B, S ⊂ B × C be two relations. Then composition of the relations R and S is denoted by SoR ⊂ A × C and is defined by (a, c) ∈ (SoR) iff b ∈ B such that (a, b) ∈ R, (b, c) ∈ S.

5. **Relations in a set** - let A (≠ φ) be a set and R ⊂ A × A i.e. R is a relation in the set A.

6. **Reflexive Relations** - R is a reflexive relation if (a, a) ∈ R, ∀ a ∈R it should be noted that if for any a ∈A such that a ʀ̸ a. then R is not reflexive.

7. **Symmetric Relation** - R is called symmetric relation on A if (x, y) ∈R ⇒ (y, x) ∈R.
   i.e. if x is related to y, then y is also related to x.
   It should be noted that R is symmetric iff R^{-1} = R.

8. **Anti Symmetric Relations** - R is called an anti symmetric relation if (a, b) ∈R and (b, a) ∈R ⇒ a = b.
   Thus if a ≠ b then a may be related to b or b may be related to a but never both.

9. **Transitive Relations** - R is called a transitive relation if (a, b) ∈R (b, c) ∈R ⇒ (a, c) ∈R

10. **Identity Relations** - R is an identity relation if (a, b) ∈R iff a = b. i.e. every element of A is related to only itself and always identity relation is reflexive symmetric and transitive.

11. **Equivalence Relations** - a relation R in a set A is called an equivalence relation if
    (i) R is reflexive i.e. (a, a) ∈R ∀ a ∈ A
    (ii) R is symmetric i.e. (a, b) ∈R ⇒ (b, a) ∈R
    (iii) R is transitive i.e. (a, b), (b, c) ∈R ⇒ (a, c) ∈R.

12. **Functions** - Suppose that to each element in a set A there is assigned, by some rule, an unique element of a set B. Such rules are called functions. If we let f denote these rules, then we write f : A → B as f is a function of A into B.

13. **Equal Functions** - If f and g are functions defined on the same domain A and if f (a) = g (a) for every a ∈A, then f = g.

14. **Constant Functions** - Let $f : A \to B$. If $f(a) = b$, a constant, for all $a \in A$, then f is called a constant function. Thus f is called a constant function if range f consists of only one element.

15. **Identity Functions** - A function f is such that $A \to A$ is called an identity function if $f(x) = x$, $\forall \, x \in A$ it is denoted by I_A.

16. **One-One Functions (Injective)** - Let $f : A \to B$ then f is called a one-one function. If no two different elements in A have the same image i.e. different elements in A have different elements in B.
    Denoted by symbol f is one-one if
       $f(a) = f(a') \Rightarrow a = a'$
    i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$
    A mapping which is not one-one is called many one function.

17. **Onto functions (Surjective)** - In the mapping $f : A \to B$, if every member of B appears as the image of atleast one element of A, then we say "f is a function of A onto B or simply f is an onto functions" Thus f is onto iff $f(A) = B$
    i.e.   range = codomain
    A function which is not onto is called into function.

18. **Inverse of a function** - Let $f : A \to B$ and $b \in B$ then the inverse of b i.e. f^{-1} (b) consists of those elements in A which are mapped onto b i.e. f^{-1} (b) = {x ; x $\in$ A, f(x) $\in$ b}
    $\therefore$ f^{-1} (b) $\subset$ A, f^{-1} (b) may be a null set or a singleton.

19. **Inverse Functions** - Let $f : A \to B$ be a one-one onto-function from A onto B. Then for each $b \in$ B. f^{-1} (b) $\in$ A and is unique. So, f^{-1} : B $\to$ A is a function defined by f^{-1} (b) = a, iff f(a) = b.
    Then f^{-1} is called the inverse function of f. If f has inverse function, f is also called invertible or non-singular.
    Thus f is invertible (non-singular) iff it is one-one onto (bijective) function.

20. **Composition Functions** - Let $f : A \to B$ and $g : B \to C$, be two functions,
    Then composition of f and g denoted by gof : $A \to C$ is defined by (gof) (a) = g {f(a)}.

21. **Binary Operation** - A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $*$ (a, b) by a $*$ b

22. **Commutative Binary Operation** - A binary operation $*$ on the set A is commutative if for every a, b $\in$ A, a $*$ b = b $*$ a.

23. **Associative Binary Operation** - A binary operation $*$ on the set A is associative if (a $*$ b) $*$ c = a $*$ (b $*$ c).

24. **An Identity Element e for Binary Operation** - Let $* : A \times A \to A$ be a binary operation. There exists an element e $\in$ A such that a $*$ e = a = e $*$ a $\forall$ a $\in$ A, then e is called an identity element for Binary Operation $*$.

25. **Inverse of an Element a** - Let $* : A \times A \to A$ be a binary operation with identity element e in A. an element a $\in$ A is invertible w.r.t. binary operation $*$, if there exists an element b in A such that a $*$ b = e = b $*$ a. and b is called the inverse of a and is denoted by a^{-1}.

CONNECTING CONCEPTS

1. In general gof $\neq$ fog.

2. $f : A \to B$, be one-one, onto then
   f^{-1} of = I_A and fof^{-1} = I_B

3. $f : A \to B$, $g : B \to C$, $h : C \to D$
   then (hog) of = ho (gof).

4. $f : A \to B$, $g : B \to C$ be one-one and onto then gof : $A \to C$ is also one-one onto and (gof)$^{-1}$ = f^{-1} o g^{-1}.

5. Let : $A \to B$, then I_B of = f and foI_A = f. It should be noted that foI_B is not defined since for
   (foI_B) (x) = fo {I_B (x)} = f(x)
   I_B (x) exist when x $\in$ B and f(x) exist when x $\in$ A

6. $f : A \to B$, $g : B \to C$ are both one-one, then gof : $A \to C$ is also one-one it should be noted that for gof to be one-one f must be one-one.

7. If $f : A \to B$ $g : B \to C$ are both onto then gof must be onto. However, the converse is not true. But for gof to be onto g must be onto.

8. The domain of the functions
$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)\,g(x)$$

is given by (dom. f) $\cap$ (dom g) while domain of the function (f/g) (x) = $\dfrac{f(x)}{g(x)}$ is given by.

(dom f) $\cap$ (dom. g) $- \{x : g(x) = 0\}$

9. If O (A) = m, O (B) = n, then total number of mappings from A to B is n^m.

10. If A and B are finite sets and O (A) = m, O (B) = n, m $\le$ n.

Then number of injection (one-one) from A to B is $^nP_m = \dfrac{n!}{(n-m)!}$

11. If f : A $\to$ B is injective (one-one), then O(A) $\le$ O (B).

12. If f : A $\to$ B is surjective (onto), then O (A) $\ge$ O (B).

13. If f : A $\to$ B is bijective (one-one onto), then O (A) = O (B).

14. Let f : A $\to$ B and O (A) = O (B), then f is one-one $\Leftrightarrow$ it is onto.

15. Let f : A $\to$ B and X_1, $X_2 \subseteq$ A, then f is one-one iff $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$

16. Let f : A $\to$ B and X $\subseteq$ A, Y $\subseteq$ B, then in general $f^{-1}(f(x)) \subseteq X$, $f(f^{-1}(y)) \subseteq Y$
If f is one-one onto $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = Y$.

2 INVERSE TRIGONOMETRIC FUNCTIONS

1. INTRODUCTION TO INVERSE TRIGONOMETRY

The inverse trigonometric functions are the inverse functions of the trigonometric functions. They are sometimes referred to as cyclometric functions.

2. IMPORTANT DEFINITIONS

Given two non-empty sets X and Y, let f:X $\rightarrow$ Y be a function, such that y = f(x). The set X is called as the domain of f while the set Y is called as the co-domain of f. The set {f(x): x $\in$ X} is called as range of f.A map f: A $\rightarrow$ B is said to be one-one or injective, if and only if, distinct elements of A have distinct images in B, i.e. if, and only if, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for all $x_1, x_2 \in A$

Onto map or Surjective map: A map f: A $\rightarrow$ B is said to be an onto map or Surjective map if, and only if, each element of B is the image of some element of A, i.e. if, and only if, Range of f = co-domain of f.

Objective map: A map f : A $\rightarrow$ B is an objective map if, and only if, it is both one – one and onto.

3. INVERSE FUNCTIONS

If f : X $\rightarrow$ Y is one-to-one and onto (i.e. f is objective), then, we can define a unique function g : Y $\rightarrow$ X, such that g(y) = x, where x $\in$ X is such that y = f (x). Thus, the domain of g = range of f and range of g = domain of f. The function is called the inverse of f and is denoted by f^{-1}.

(a) Trigonometric functions are many-one functions but these become one-one, onto, if we restrict the domain of trigonometric functions. Similarly, co-domain is equated to range to make it an onto function. We can say that the inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions.

(b) Inverse of sin (sine functions) is denoted by sin^{-1} (arc sine function). We also write it as sin^{-1} x. Similarly, other inverse trigonometric functions are given by cos^{-1} x, tan^{-1} x, sec^{-1} x, cot^{-1}x and cosec^{-1} x.

(c) Note that $\sin^{-1} x \neq \dfrac{1}{\sin x}$ and $(\sin^{-1} x)^2 \neq \sin^{-2} x$, Also $\sin^{-1} x \neq (\sin x)^{-1}$

(d) Domain and Range of Inverse Trigonometric Functions:

	Function	Domain	Range (Principal value branch)
(i)	$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$

	Function	Domain	Range (Principal value branch)
(ii)	$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii)	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} \leq y < \dfrac{\pi}{2}$
(iv)	$y = \csc^{-1} x$	$x \geq 1$ or $x \leq -1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}, y \neq 0$
(v)	$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$
(vi)	$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

(e) The principal value of an inverse trigonometric function is the value of that inverse trigonometric function which lies in the range of principal branch.

MASTERJEE CONCEPTS

If no branch of an inverse trigonometric function is mentioned, then it can be implied that the principal value branch of that function.

You can remember range as set of angles that have the smallest absolute values satisfying for all the values of domain.

Vaibhav Gupta (JEE 2009 AIR 54)

4. TRANSFORMATION OF TRIGONOMETRIC FUNCTIONS TO INVERSE TRIGONOMETRIC FUNCTIONS

4.1 sin x to sin⁻¹ x

The graph of an inverse trigonometric function can be obtained from the graph of the original by interchanging x and y axes.

Note: It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as the mirror image in the line y = x.
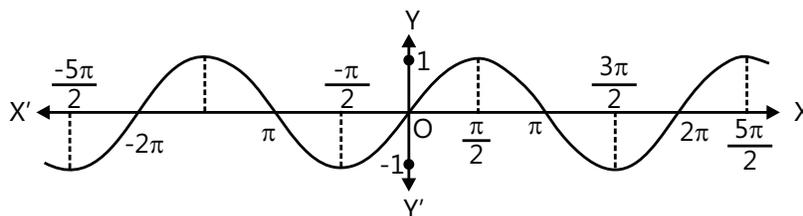

Figure 20.1

(a) $y = \sin x$, $x \in R$ and $|y| \leq 1$; $y = \sin^{-1} x, |x| \leq 1, y \in [-\pi/2, \pi/2]$

4.2 cos x to cos⁻¹ x

(b) $y = \cos x, x \in R \text{ and } |y| \leq 1$ $y = \cos^{-1} x, x \in [-1,1] \text{ and } y \in [0, \pi]$

y-sin x and y-sin⁻¹(x)

y-sin⁻¹(x)

Figure 20.2

y=cos x
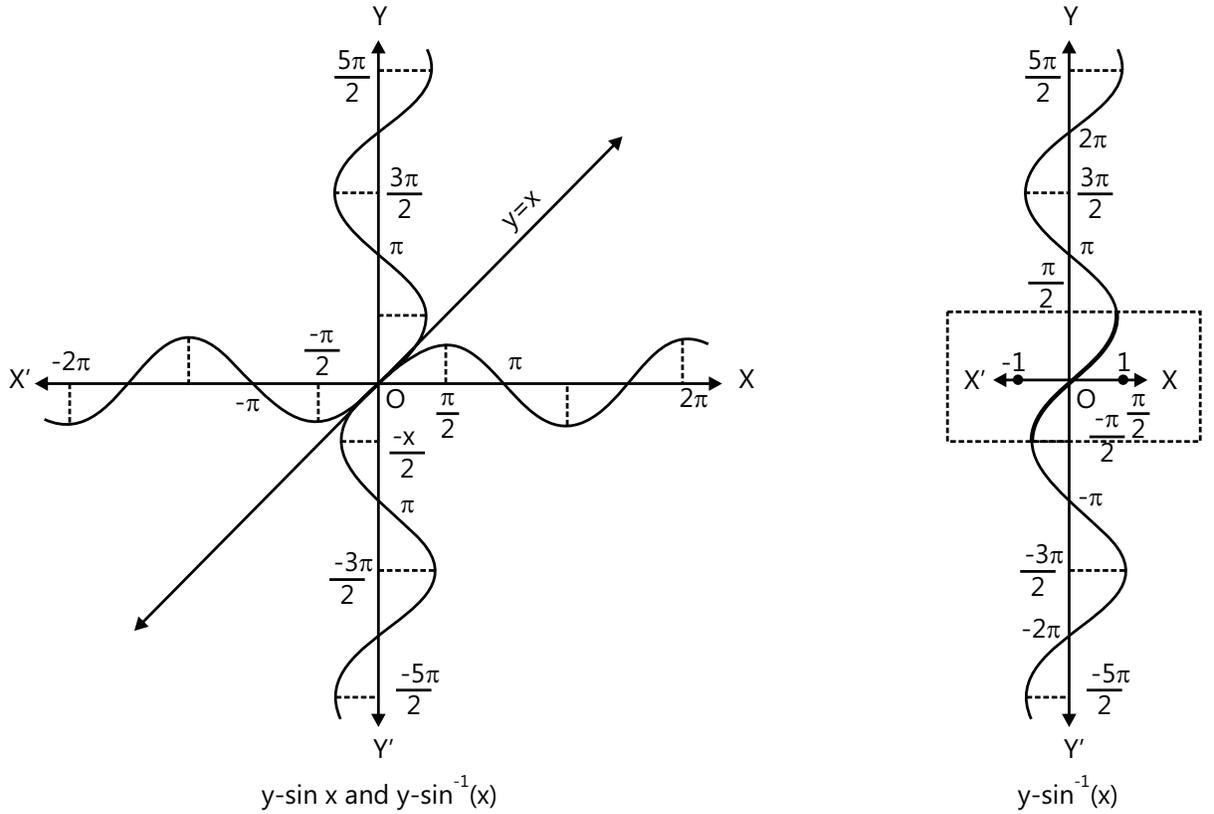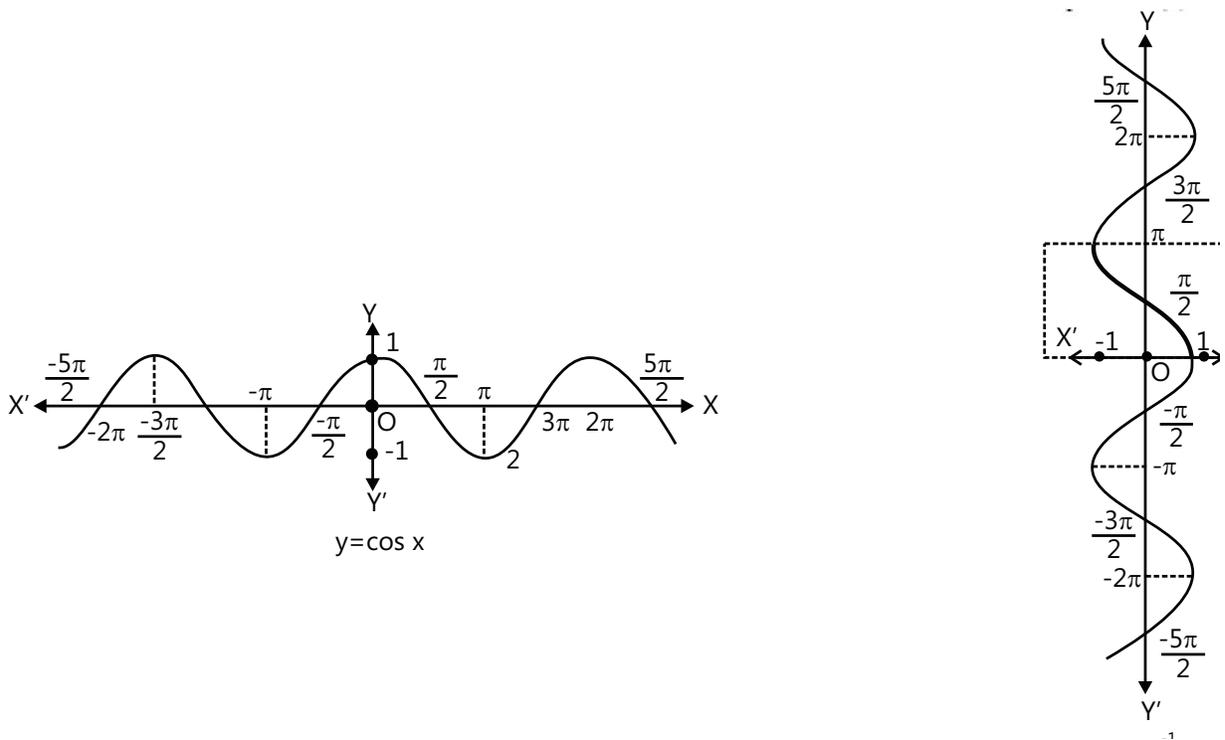
4.3 tan x to tan^{-1} x

(c) $y = \tan x, x \in R - \left\{ x : x = (2n+1)\dfrac{\pi}{2}, n \in Z \right\}$ and $y \in R$ $y = \tan^{-1} x, x \in R$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right)$
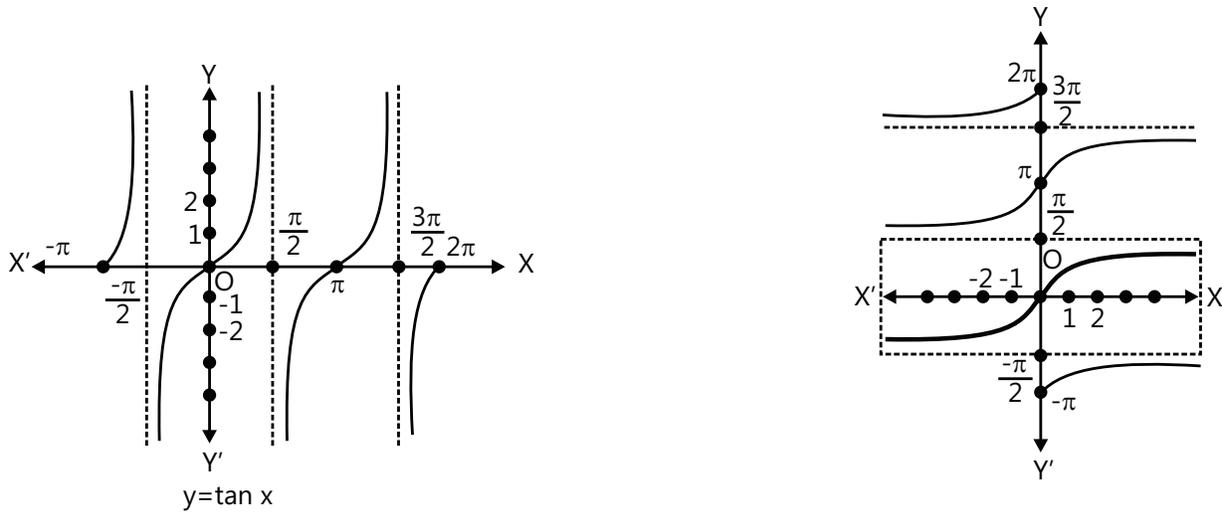

y=tan x

Figure 20.4

4.4 cot x to cot^{-1} x

(d) $y = \cot x, x \in R - \left\{ x : x = n\pi, \, n \in Z \right\}$ and $y \in R$ $y = \cot^{-1} x, x \in R$ and $y \in (0, \pi)$

y=tan x

y=tan^{-1} x

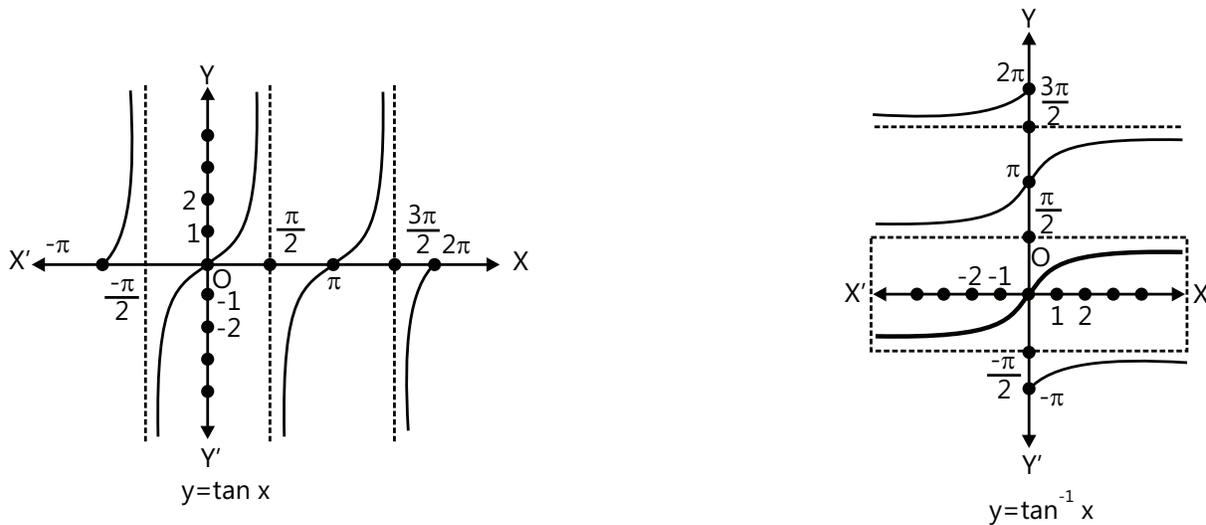**Figure 20.5**

4.5 sec x to sec^{-1} x

(e) $y = \sec x, x \in R - \left\{ x : x = (2n+1)\dfrac{\pi}{2}, n \in z \right\}$ and $y \in R - (-1,1)$ $y = \sec^{-1} x, x \in R - (-1,1)$ and $y \in [0, \pi] \left\{ \dfrac{\pi}{2} \right\}$
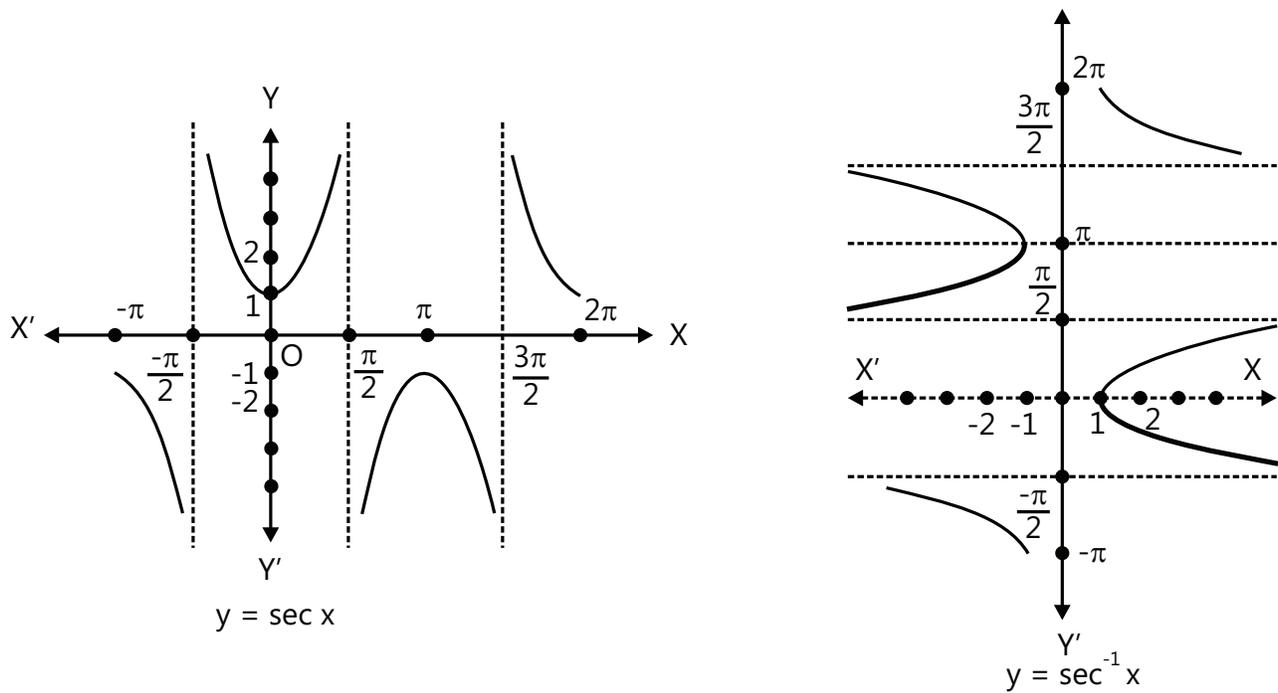
y = sec x

y = sec⁻¹ x

Figure 20.6

4.6 cosec x to cosec⁻¹ x

(f) $y = \operatorname{cosec} x, x \in \{x : x = n\pi, n \in Z\}$ and $y \in R - (-1,1)$ $y = \operatorname{cosec}^{-1} x, x \in R - (-1,1)$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
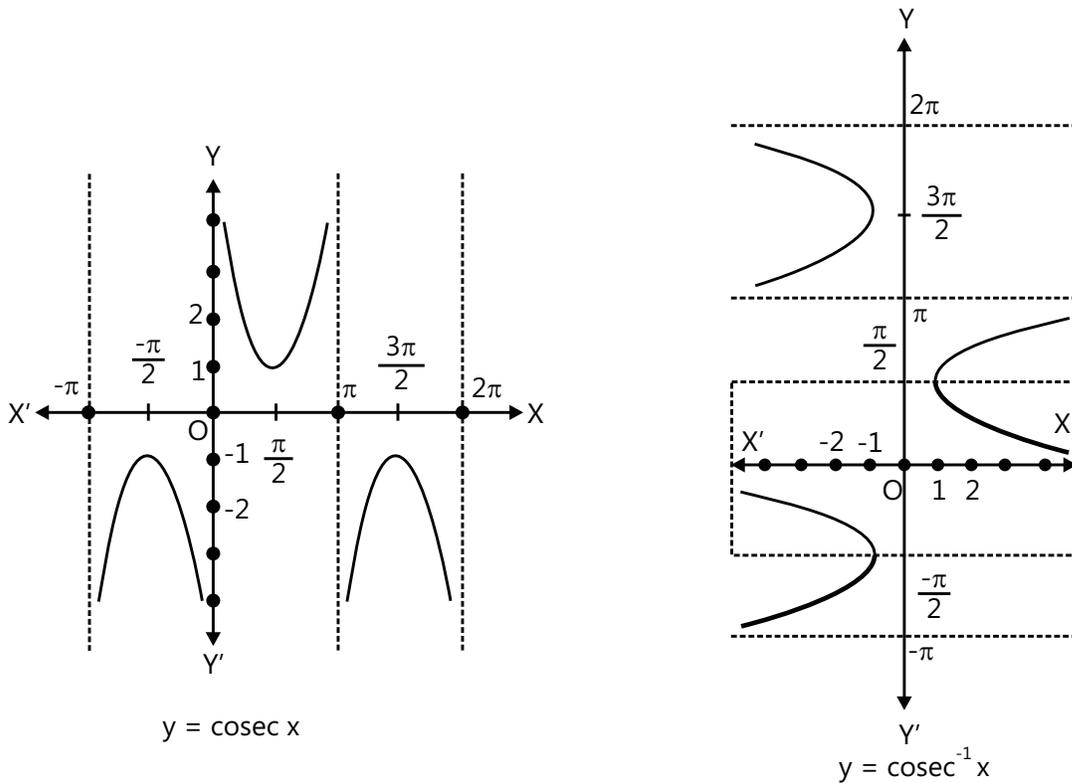

y = cosec x

y = cosec⁻¹ x

Illustration 1: Find the domain of definition of the function f(x) = $\sqrt{3\cos^{-1}(4x) - \pi}$. **(JEE MAIN)**

Sol: Use the condition that the expression inside the square root is ≥ zero.

For domain of f(x) = $\sqrt{3\cos^{-1}(4x) - \pi}$, we must have

$$4x \geq \cos\left(\frac{\pi}{3}\right) \Rightarrow 4x \geq \frac{1}{2} \Rightarrow x \geq \frac{1}{8}$$ .....(i)

Also $-1 \leq 4x \leq 1 \Rightarrow \frac{-1}{4} \leq x \leq \frac{1}{4}$ .....(ii)

∴ From (i) and (ii), we get $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

MASTERJEE CONCEPTS

In case of confusion, try solving problems by replacing inverse functions with angles and applying trigonometric identities.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 2: If $0 < \cos^{-1}x < 1$ and $1 + \sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + \sin^3(\cos^{-1}x) + ...\infty = 2$, then find the value of x. **(JEE MAIN)**

Sol: Use summation of infinite GP series.

We have $1 + \sin(\cos^{-1}x) + \sin^2(\cos^{-1}x) + ... \infty = 2$

$$\Rightarrow \frac{1}{1 - \sin(\cos^{-1}x)} = 2 \Rightarrow \frac{1}{2} = 1 - \sin(\cos^{-1}x) \Rightarrow \sin(\cos^{-1}x) = \frac{1}{2} \Rightarrow \cos^{-1}x = \frac{\pi}{6} \Rightarrow x = \frac{\sqrt{3}}{2}$$

Illustration 3: Let $f(x) = \frac{2}{\pi}(\sin^{-1}[x] + \tan^{-1}[x] + \cot^{-1}[x])$, where [x] denotes the greatest integer less than or equal to x. If A and B denote the domain and range of f(x) respectively, find the number of integers in A∪B. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}[x] + \cot^{-1}[x] = \frac{\pi}{2}$ and proceed.

For domain of f(x), we must have $-1 \leq [x] \leq 1 \Rightarrow -1 \leq x < 2$, so set A = [-1,2)

$$f(x) = \frac{2}{\pi}\left(\sin^{-1}[x] + \frac{\pi}{2}\right)$$ $$\left(\text{As} \quad \tan^{-1}[x] + \cot^{-1}[x] = \frac{\pi}{2}, \forall x \in A\right)$$

So, set B = {0, 1, 2} = Range of f(x). Now, $A \cup B = [-1,2) \cup \{0,1,2\} = [-1,2]$

Hence, number of integers in $(A \cup B) = 4$

5. PROPERTIES/IDENTITIES OF INVERSE TRIGONOMETRIC FUNCTIONS

5.1 Complementary Angles

(a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1,1]$

(b) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}, \forall x \in R$

(c) $\sec^{-1}x + \csc^{-1}x = \dfrac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

5.2 Negative Arguments

(a) $\sin^{-1}(-x) = -\sin^{-1}x, \ \forall x \in [-1, 1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \ \forall x \in [-1, 1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, \forall x \in R$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall x \in R$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(f) $\csc^{-1}(-x) = -\csc^{-1}x \forall x \in (-\infty, -1) \cup [1, \infty)$

5.3 Reciprocal Arguments

(a) $\csc^{-1}x = \sin^{-1}\dfrac{1}{x}; |x| \geq 1$ (Both the functions are identical)

and $\sin^{-1}x = \csc^{-1}\dfrac{1}{x}; |x| \leq 1, x \neq 0$ (Both the functions are not identical)

(b) $\sec^{-1}x = \cos^{-1}\dfrac{1}{x}; |x| \geq 1$ (Both the functions are identical)

and $\cos^{-1}x = \sec^{-1}\dfrac{1}{x}; |x| \leq 1$ (Both the functions are not identical)

(c) $\tan^{-1}x = \cot^{-1}\left(\dfrac{1}{x}\right), \quad x \in (0, \infty) = -\pi + \cot^{-1}\left(\dfrac{1}{x}\right), \quad x \in (-\infty, 0),$

and $\cot^{-1}x = \tan^{-1}\left(\dfrac{1}{x}\right), \quad x \in (0, \infty) = \pi + \tan^{-1}\left(\dfrac{1}{x}\right), \quad x \in (-\infty, 0)$

5.4 Forward Inverse Identities

(a) $y = \sin(\sin^{-1}x) = x, x \in [-1, 1], y \in [-1, 1],$ y is aperiodic

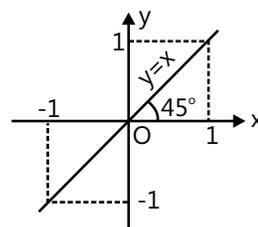**(b)** $y = \cos(\cos^{-1}x) = x, x \in [-1, 1], y \in [-1, 1],$ y is aperiodic
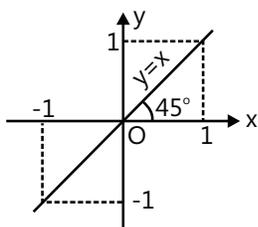
Figure 20.8

(c) $y = \tan(\tan^{-1}x) = x$, $x \in R$, $y \in R$, y is aperiodic   **(d)** $y = \cot(\cot^{-1}x) = x$, $x \in R$, $y \in R$, y is aperiodic

Figure 20.9

(e) $y = \operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$, $|x| \geq 1, |y| \geq 1$, y is aperiodic   **(f)** $y = \sec(\sec^{-1}x) = x$, $|x| \geq 1, |y| \geq 1$, y is aperiodic
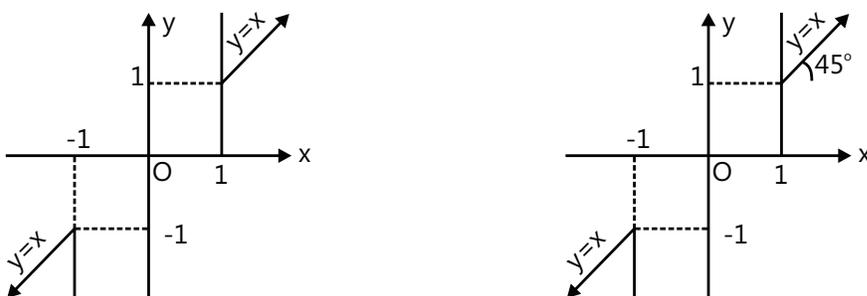
Figure 20.10

Also,

$$\cos(\sin^{-1}x) = \sqrt{1-x^2} \qquad\qquad\qquad \sin(\cos^{-1}x) = \sqrt{1-x^2}$$

$$\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}} \qquad\qquad \tan(\cos^{-1}x) = \frac{\sqrt{1-x^2}}{x}$$

$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}} \qquad\qquad \tan\left(\sin^{-1}x\right) = \frac{x}{\sqrt{1-x^2}}$$

5.5 Inverse Forward Identities

(a) $y = \sin^{-1}(\sin x) = x, \lceil x \in R \rceil, y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$, Periodic with period 2π
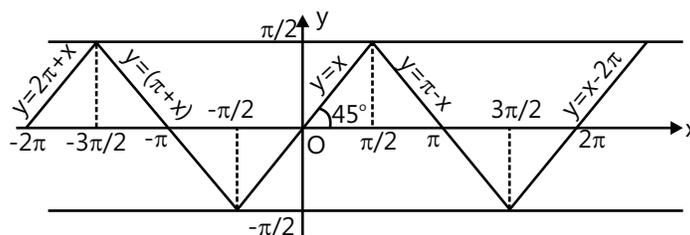
(b) $y = \cos^{-1}(\cos x) = x, x \in R, y \in [0, \pi]$, Periodic with period 2π
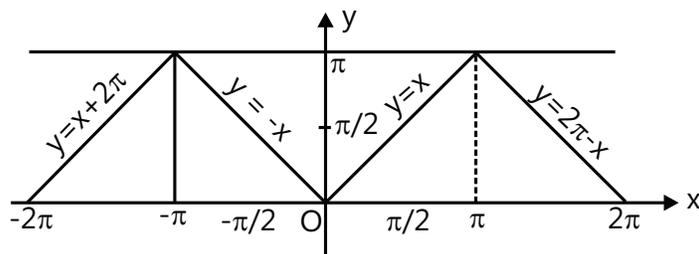
Figure 20.12

(c) $y = \tan^{-1}(\tan x) = x, x \in R - \left\{(2n-1)\dfrac{\pi}{2} n \in I\right\}, y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, Periodic with period π

Figure 20.13

(d) $y = \cot^{-1}(\cot x) = x, x \in R - \{n\pi\}, y \in (0, \pi)$, periodic with π

Figure 20.14

(e) $y = \mathrm{cosec}^{-1}(\mathrm{cosec}\, x) = x \in R - \{n\pi, n \in I\}, y \in \left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$ y is periodic with period 2π

(f) $y = \sec^{-1}(\sec x) = x$, y is periodic,
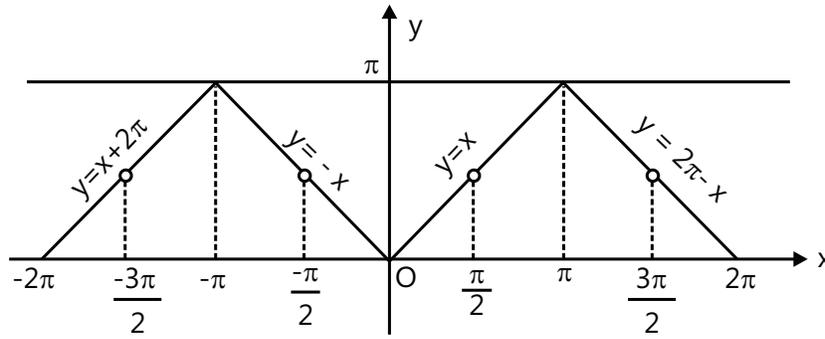
Figure 20.16

$x \in R - \left\{(2n-1)\dfrac{\pi}{2} n \in I\right\}, y \in \left[0, \dfrac{\pi}{2}\right] \cup \left(\dfrac{\pi}{2}, \pi\right]$ with period 2π

(i) $\tan^{-1}(\cot x) = \dfrac{1}{2}\pi - x$ for $x \in [0, \pi]$

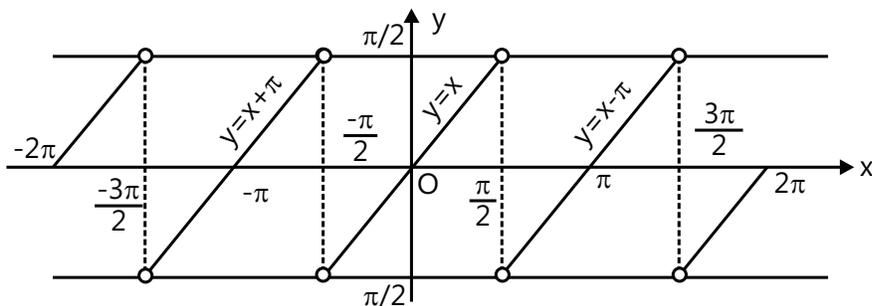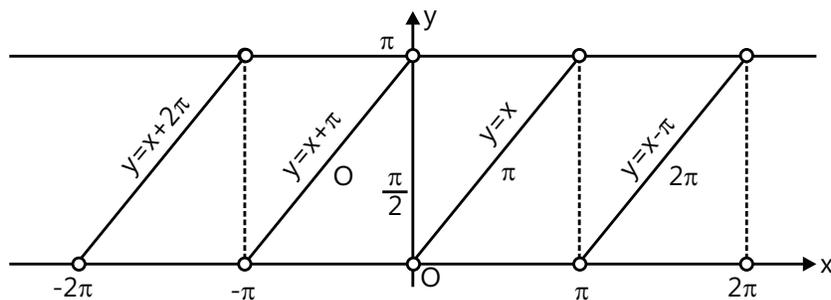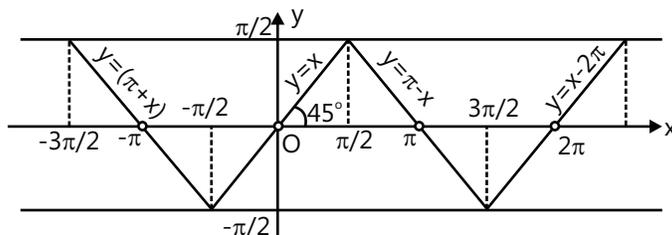**(ii)** $\sin^{-1}(\operatorname{cosec} x) = \dfrac{1}{2}\pi - x$ for $x \in [0, \pi]$

(iii) $\sec^{-1}(\cos x) = \dfrac{1}{2}\pi - x$ for $x \in \left[0, \dfrac{1}{2}\pi\right]$.

5.6 Sum of Angles

(a) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) & \text{if } x \geq 0; y \geq 0 \text{ and } x^2 + y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \; x > 0; y > 0$

(c) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y > 0 and $x^2 + y^2 \leq 1$

(d) $\cos^{-1} x \pm \cos^{-1} y = \pi - \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$ if x, y > 0 and $x^2 + y^2 \leq 1$

(e) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy} & x > 0 \; y > 0 \text{ and } xy < 1 \Rightarrow 0 < \tan^{-1}x + \tan^{-1}y < \dfrac{\pi}{2} \\ \pi - \tan^{-1}\dfrac{x+y}{1-xy} & x > 0 \; y > 0 \text{ and } xy > 1 \Rightarrow \dfrac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi \end{cases}$

(f) x > 0 & y > 0 then $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$ (with no other restriction)

(g) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$; $\tan^{-1} - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Illustration 4: Evaluate: $\sin\left(\tan^{-1}\dfrac{15}{8}\right)$ (JEE MAIN)

Sol: Convert $\tan^{-1}\dfrac{15}{8}$ to $\sin^{-1}$.

We know that $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$, So, will convert each expression in the form $\sin(\sin^{-1} x)$ by using

$$\cos^{-1}\frac{b}{h} = \sin^{-1}\frac{p}{h}, \quad \tan^{-1}\frac{p}{b} = \sin^{-1}\frac{p}{h}, \cot^{-1}\frac{p}{b} = \sin^{-1}\frac{b}{h} \text{ etc.}$$

Where b, p and h denote the base, perpendicular and hypotenuse of a right triangle.

h=17   p=15   b=8
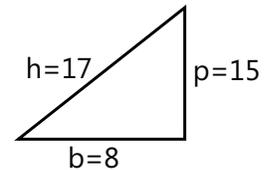
Figure 20.17

$$\sin\left(\tan^{-1}\frac{15}{8}\right) = \sin\left(\sin^{-1}\frac{15}{17}\right) = \frac{15}{17}$$

Illustration 5: Evaluate: $\cos\left(\operatorname{cosec}^{-1}\dfrac{13}{12}\right)$ (JEE MAIN)

Sol: Write $\operatorname{cosec}^{-1}$ in terms of $\cos^{-1}$.

$$\cos\left(\operatorname{cosec}^{-1}\frac{13}{12}\right) = \cos\left(\cos^{-1}\frac{5}{13}\right) = \frac{5}{13}\left[\because \operatorname{cosec}^{-1}\frac{13}{12} = \cos^{-1}\frac{5}{13}\right]$$

h=13   p=12   b=5
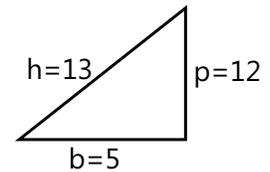
Figure 20.18

Illustration 6: Find the principal value of $\cot^{-1}(-\sqrt{3})$ (JEE MAIN)

Sol: The principal value of $\cot^{-1} x$ lies in between 0 to π.

Let $\cot^{-1}(-\sqrt{3}) = \theta$

Then $\cot\theta = -\sqrt{3} = -\cot\dfrac{\pi}{6}$

Since principal value branch of $\cot^{-1} x$ is $0 < \theta < \pi$. Therefore, we want to find the value of θ such that $0 < \theta < \pi$.

Now, $\cot\theta = -\cot\dfrac{\pi}{6} = \cot\left(\pi - \dfrac{\pi}{6}\right) = \cot\dfrac{5\pi}{6}$

Therefore, principal value of $\cot^{-1}(-\sqrt{3}) = \dfrac{5\pi}{6}$.

Illustration 7: $\sin^{-1}\left(\sin\dfrac{10\pi}{7}\right) = \dfrac{10\pi}{7}$ <div style="text-align:right">**(JEE MAIN)**</div>

Sol: Write $\dfrac{10\pi}{7}$ as $\pi + \dfrac{3\pi}{7}$ and expand.

$= \sin^{-1}\left(\sin\dfrac{10\pi}{7}\right) = \sin^{-1}\left(-\sin\left(\dfrac{3\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(-\dfrac{3\pi}{7}\right)\right) = -\dfrac{3\pi}{7}$

Illustration 8: $\cos^{-1}\left(\sin\left(-\dfrac{\pi}{9}\right)\right)$ <div style="text-align:right">**(JEE MAIN)**</div>

Sol: $= \cos^{-1}\left(\cos\left(\dfrac{\pi}{2} + \dfrac{\pi}{9}\right)\right) = \cos^{-1}\left(\cos\left(\dfrac{11\pi}{18}\right)\right) = \dfrac{11\pi}{18}$

Illustration 9: $\sin^{-1}\left(\cos\dfrac{13\pi}{10}\right)$ <div style="text-align:right">**(JEE MAIN)**</div>

Sol: Similar to previous example.

$= \sin^{-1}\cos\dfrac{13\pi}{10} = \sin^{-1}\left(-\cos\dfrac{3\pi}{10}\right) = \sin^{-1}\left(-\sin\left(\dfrac{5\pi}{10} - \dfrac{2\pi}{10}\right)\right) = \sin^{-1}\left(-\sin\dfrac{\pi}{5}\right) = \sin^{-1}\left(\sin\left(-\dfrac{\pi}{5}\right)\right) = -\dfrac{\pi}{5}$

Illustration 10: Find the principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$. <div style="text-align:right">**(JEE MAIN)**</div>

Sol: Let $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = y$. Then $\sin y = \dfrac{1}{\sqrt{2}} \Rightarrow y = \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

We know that, the range of the principal value branch of $\sin^{-1}$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.

Therefore, principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ is $\dfrac{\pi}{4}$.

Illustration 11: Find the integral solution of the inequality $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$. <div style="text-align:right">**(JEE ADVANCED)**</div>

Sol: Use inverse forward identities to simplify the equation.

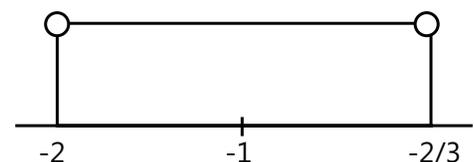|	
$3x^2 + 8x < -4$	$\Rightarrow \quad 3x^2 + 8x + 4 < 0$
$\Rightarrow \quad 3x^2 + 6x + 2x + 4 < 0 \Rightarrow$	$3x(x+2) + 2(x+2) < 0$
$(x+2)(3x+2) < 0$	$x = -1$

Figure 20.19

Illustration 12: Find the largest integral value of k, for which $(k-2)x^2 + 8x + k + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$, for all $x \in R$. <div style="text-align:right">**(JEE ADVANCED)**</div>

Sol: Use inverse forward identities.

$\sin^{-1}(\sin 12) = \sin^{-1}(\sin(12 - 4\pi)) = 12 - 4\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

$\therefore (k-2)x^2 + 8x + k + 4 > 0, \quad \forall x \in R$

If k = 2, then 8x+4>0, (not possible)

and if $k \neq 2$, then k-2>0 $\Rightarrow$ k>2

and $64 - 4(k-2)(k+4) < 0 \Rightarrow 16 < k^2 + 2k - 8$

$\Rightarrow k^2 + 2k - 24 > 0 \Rightarrow (k+6)(k-4) > 0$

K = 5
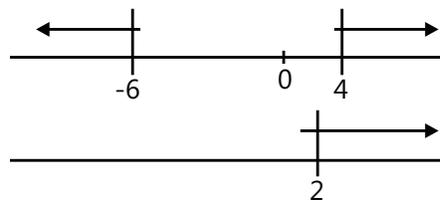
Figure 20.20

Illustration 13: Find domain of $f(x) = \dfrac{1}{\sqrt{\ln\left(\cot^{-1} x\right)}}$. **(JEE MAIN)**

Sol: Find the range of x for which $\ln(\cot^{-1}x) > 0 \Rightarrow \cot^{-1}x > 1 \Rightarrow x < \cot 1 \Rightarrow x \in (-\infty, \cot 1)$

Illustration 14: Evaluate the following: **(JEE MAIN)**

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right)$ (iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ (iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\pi}{6}\right\}$

Sol: Recall that, $\sin^{-1}(\sin\theta) = \theta$, if $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}, \cos^{-1}(\cos\theta) = \theta$, if $0 \leq \theta \leq \pi$ and

$\tan^{-1}(\tan\theta) = \theta$, if $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Therefore,

(i) $\sin^{-1}\left(\sin\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$ (ii) $\tan^{-1}\left(\tan\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$

(iii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) \neq \dfrac{7\pi}{6}$, because $\dfrac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \dfrac{5\pi}{6}\right)\right)$ $\left[\because \dfrac{7\pi}{6} = 2\pi - \dfrac{5\pi}{6}\right] = \cos^{-1}\left(\cos\dfrac{5\pi}{6}\right)$ $[\because \cos(2\pi - \theta) = \cos\theta] = \dfrac{5\pi}{6}$

(iv) $\cos\left\{\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\pi}{6}\right\} = \cos\left(\dfrac{5\pi}{6} + \dfrac{\pi}{6}\right)$ $\left[\because \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}\right]$

Illustration 15: Evaluate the following:

(i) $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$ (ii) $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ (iii) $\sin(\cot^{-1} x)$ **(JEE MAIN)**

Sol: (i) Let $\cos^{-1}\dfrac{3}{5} = \theta$. Then, $\cos\theta = \dfrac{3}{5} \Rightarrow \sin\theta = \dfrac{4}{5}$

$\therefore \sin\left(\cos^{-1}\dfrac{3}{5}\right) = \sin\theta = \dfrac{4}{5}$

(ii) $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right) = \sin\left(\dfrac{\pi}{2} - \left(-\dfrac{\pi}{6}\right)\right) = \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$

(iii) Let $\cot^{-1} x = \theta$, Then, $x = \cot\theta$

Now, $\cot\theta = x \Rightarrow \sin\theta = \dfrac{1}{\sqrt{1+x^2}}$ $\therefore \sin(\cot^{-1}x) = \sin\theta = \dfrac{1}{\sqrt{1+x^2}}$

Illustration 16: Evaluate the following:

(i) $\sin^{-1}(\sin5)$ (ii) $\cos^{-1}(\cos10)$ **(JEE MAIN)**

Sol: Notice that the angle is in radians.

(i) Here, $\theta = 5$ radians. Clearly, it does not lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. But

$2\pi - 5$ and $5 - 2\pi$ both lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ such that

$\sin(5 - 2\pi) = \sin(-(2\pi - 5)) = -\sin(2\pi - 5) = -(-\sin5) = \sin5$

$\Rightarrow \sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi.$

(ii) We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$. Here, $\theta = 10$ radians. Clearly, it does not lie between 0 and π such that, $(4\pi - 10) = \cos10 \Rightarrow \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$

Illustration 17: Evaluate the following:

(i) $\sin^{-1}(2\sin^{-1}0.8)$ (ii) $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right)$ **(JEE MAIN)**

Sol: Write the term inside the brackets in (i) and (ii) as $\sin^{-1}$ and $\tan^{-1}$ respectively.

(i) We know that: $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

$\therefore 2\sin^{-1}0.8 = \sin^{-1}(2 \times 0.8 \times \sqrt{1 - 0.64})$

$\Rightarrow \sin^{-1}(2\sin^{-1}0.8) = \sin\{\sin^{-1}(0.96)\} = 0.96$

(ii) $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right)$

$= \tan\left(\tan^{-1}\dfrac{5}{12} - \dfrac{\pi}{4}\right)$ $\left[\text{From (ii) we have, } 2\tan^{-1}\dfrac{1}{5} = \tan^{-1}\dfrac{5}{12}\right]$

$= \tan\left(\tan^{-1}\dfrac{5}{12} - \tan^{-1}1\right)$ $\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) \text{ if } xy > -1\right]$ $= \tan\left[\tan^{-1}\left(\dfrac{-7}{17}\right)\right] = -\dfrac{7}{17}$

Illustration 18: Write the following in their simplest forms:

(i) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ (ii) $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$ **(JEE ADVANCED)**

Sol: (i) Use the formula $1 - \cos x = 2\sin^2 x/2$ and $1 + \cos x = 2\cos^2 x/2$

(ii) Write the term inside the square bracket in terms of $\sin^{-1}$.

(i) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}=\tan^{-1}\sqrt{\dfrac{2\sin^2 x/2}{2\cos^2 x/2}}=\tan^{-1}\left|\tan\dfrac{x}{2}\right|=\dfrac{|x|}{2}$

(ii) $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$

$=\sin\left[\cot^{-1}\left\{\cos\left(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)\right\}\right]\ =\sin\left(\cot^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)=\sin\left\{\sin^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\}\quad\left[\because\cot^{-1}x=\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right]$

$=\sqrt{\dfrac{1+x^2}{2+x^2}}$

Illustration 19: Express $\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right),\ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ in the simplest form. **(JEE ADVANCED)**

Sol: Convert the term inside the bracket in terms of $\tan\dfrac{x}{2}$ and proceed.

We write, $\Rightarrow\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right)=\tan^{-1}\left[\dfrac{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2}+\sin^2\dfrac{x}{2}-2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}\right]$

$=\left[\dfrac{\left(\cos\dfrac{x}{2}+\sin\dfrac{x}{2}\right)\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)}{\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)^2}\right]=\tan^{-1}\left[\dfrac{\cos\dfrac{x}{2}+\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}-\sin\dfrac{x}{2}}\right]=\tan^{-1}\left[\dfrac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}\right]=\tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right]=\dfrac{\pi}{4}+\dfrac{x}{2}$

Alternatively,

$\tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right)=\tan^{-1}\left[\dfrac{\sin\left(\dfrac{\pi}{2}-x\right)}{1-\cos\left(\dfrac{\pi}{2}-x\right)}\right]=\tan^{-1}\left[\dfrac{\sin\left(\dfrac{\pi-2x}{2}\right)}{1-\cos\left(\dfrac{\pi-2x}{2}\right)}\right]$

$=\tan^{-1}\left[\dfrac{2\sin\left(\dfrac{\pi-2x}{4}\right)\cos\left(\dfrac{\pi-2x}{4}\right)}{2\sin^2\left(\dfrac{\pi-2x}{4}\right)}\right]=\tan^{-1}\left[\cot\left(\dfrac{\pi-2x}{4}\right)\right]=\tan^{-1}\left[\tan\left(\dfrac{\pi}{2}-\dfrac{\pi-2x}{4}\right)\right]=\tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right]=\dfrac{\pi}{4}+\dfrac{x}{2}$

Illustration 20: If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right)=1$, find the value of x. **(JEE MAIN)**

Sol: From the question, we have $\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right)=\dfrac{\pi}{2}$ and proceed.

We have $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right)=1$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1 \qquad \Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5} \qquad \Rightarrow \quad \cos^{-1}x=\cos^{-1}\frac{1}{5} \Rightarrow x=\frac{1}{5}$$

Illustration 21: Find the value of $\cos(\sec^{-1}x+\csc^{-1}x), |x|\geq 1$. **(JEE MAIN)**

Sol: Use $\sec^{-1}x+\csc^{-1}x=\frac{\pi}{2}$

We have $\cos\left(\sec^{-1}x+\csc^{-1}x\right)=\cos\left(\frac{\pi}{2}\right)=0$

Illustration 22: Find maximum & minimum values of $\left(\sec^{-1}x\right)^2+\left(\csc^{-1}x\right)^2$. **(JEE ADVANCED)**

Sol: Apply the identity $\sec^{-1}x+\csc^{-1}x=\frac{\pi}{2}$ and then use suitable substitution to form a quadratic.

$$y=(\sec^{-1}x)^2+(\csc^{-1}x)^2$$

$$=(\sec^{-1}x+\csc^{-1}x)^2-2\sec^{-1}x\,\csc^{-1}x$$

put $\quad t=\sec^{-1}x \qquad ; \qquad \sec^{-1}x+\csc^{-1}x=\frac{\pi}{2}$

$$y=\frac{\pi^2}{4}-2t\left(\frac{\pi}{2}-t\right)=2t^2-\pi t+\frac{\pi^2}{4}$$

$$y=2\left[t^2-\frac{\pi}{2}t+\frac{\pi^2}{8}\right]=2\left[\left(t-\frac{\pi}{4}\right)^2+\frac{\pi^2}{16}\right]=\frac{\pi^2}{8}+2\left(t-\frac{\pi}{4}\right)^2 \quad \therefore y_{min}=\frac{\pi^2}{8}; \quad y_{max}=\frac{5\pi^2}{4} \text{ at } t=\pi$$

Illustration 23: Find the range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$. **(JEE MAIN)**

Sol: Find the domain of the given function and then find the range.

$f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$

Here domain is only x = 1or –1;

So range will contain only 2 elements $\{3\pi/4, \pi/4\}$

Illustration 24: Find the number of solutions of the equation $\tan^{-1}x^3+\cot^{-1}(e^x)=\frac{\pi}{2}$. **(JEE ADVANCED)**

Sol: Use $\tan^{-1}A+\cot^{-1}A=\frac{\pi}{2}$ to simplify the given equation and then take

the help of graph to find the number of solution.

$\cot^{-1}(e^x)=\frac{\pi}{2}-\tan^{-1}(x^3)=\cot^{-1}(x^3) \Rightarrow e^x=x^3 \Rightarrow x^3e^{-x}=1$

Figure 20.21

Plotting the graph of y = I and $y=x^3e^{-x}$ we can see that the line intersects the curve at two points. Hence there are 2 solutions for the above equation.
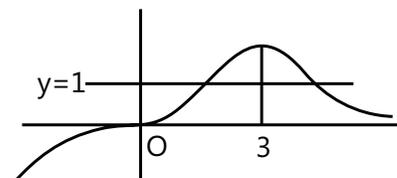
Illustration 25: Find the number of values of x satisfying the equation

$$\tan^{-1}\left(x - \frac{x^3}{4} + \frac{x^5}{16} - \ldots\ldots\right) + \cot^{-1}\left(x + \frac{x^2}{2} + \frac{x^3}{4} - \ldots\ldots\right) = \frac{\pi}{2} \text{ for } 0 < |x| < 2.$$ **(JEE ADVANCED)**

Sol: Use $\tan^{-1} A + \cot^{-1} A = \frac{\pi}{2}$.

We must have $x - \frac{x^3}{4} + \frac{x^5}{16} - \ldots\ldots = x + \frac{x^2}{2} + \frac{x^3}{4} + \ldots\ldots$

$$\Rightarrow \frac{x}{1 + \frac{x^2}{4}} = \frac{x}{1 - \frac{x}{2}} \Rightarrow \frac{4x}{4 + x^2} = \frac{2x}{2 - x} \Rightarrow 2x^2(x + 2) = 0$$

$\therefore \quad x = 0, \ -2 \quad$ (As $0 < |x| < 2$)

Clearly no value of x satisfies given equation.

Illustration 26: Prove that $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

We have, $= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left\{\dfrac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}\right\}$ $\left[\begin{array}{l}\because \tan^{-1} x + \tan^{-1} y \\ = \tan^{-1}\left(\dfrac{x + y}{1 - xy}\right) \text{ If } xy < 1\end{array}\right]$

$= \tan^{-1}\left\{\frac{48 + 77}{264 - 14}\right\} = \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Illustration 27: If $\tan^{-1} 4 + \tan^{-1} 5 = \cot^{-1}(\lambda)$ then find λ. **(JEE MAIN)**

Sol: Write the L.H.S. in terms of $\cot^{-1}$ and compare.

We have $\tan^{-1} 4 + \tan^{-1} 5 = \tan^{-1}\frac{4 + 5}{1 - 20} = \pi - \tan^{-1}\frac{9}{19} = \pi - \cot^{-1}\frac{19}{9}$

$= \cot^{-1}\left(-\frac{19}{9}\right) \qquad \Rightarrow \lambda = -\frac{19}{9}$

Illustration 28: Prove that: $\tan^{-1}\frac{1 - x}{1 + x} - \tan^{-1}\frac{1 - y}{1 + y} = \sin^{-1}\left(\frac{y - x}{\sqrt{1 + x^2}\sqrt{1 + y^2}}\right)$ **(JEE MAIN)**

Sol: Use the formula $\tan^{-1}\left(\frac{x + y}{1 - xy}\right) = \tan^{-1} x + \tan^{-1} y$.

We have, LHS $= \tan^{-1}\frac{1 - x}{1 + x} - \tan^{-1}\frac{1 - y}{1 + y} = (\tan^{-1} 1 - \tan^{-1} x) - (\tan^{-1} 1 - \tan^{-1} y) = \tan^{-1} y - \tan^{-1} x$

$$= \tan^{-1}\left(\frac{y-x}{1+yx}\right) = \sin^{-1}\left(\frac{y-x}{\sqrt{(1+yx)^2 + (y-x)^2}}\right) = \sin^{-1}\left\{\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right\} = \text{RHS}$$

Illustration 29: Prove that: (i) $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

(iii) $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$ **(JEE ADVANCED)**

Sol: Same as above.

(i) LHS $= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13}$

$= \tan^{-1}\left\{\frac{\frac{1}{7} + \frac{1}{13}}{1 - \frac{1}{7} \times \frac{1}{13}}\right\}$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy < 1\right] = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\frac{2}{9} = \text{R.H.S.}$

(ii) L.H.S. $= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$

$= \left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right) - \tan^{-1}\frac{8}{19} = \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\frac{8}{19} = \tan^{-1}\left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \times \frac{8}{19}}\right) = \tan^{-1}\frac{425}{425} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$

(iii) L.H.S. $= \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$

$= \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$ $= \tan^{-1}\left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}}\right)$

$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$ $= \tan^{-1}\left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1} = \frac{\pi}{4} = \text{R.H.S.}$

Illustration 30: Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$.   **(JEE MAIN)**

Sol: We have, L.H.S. $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2} + \frac{2}{11}}{1 - \frac{1}{2} \times \frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$

Illustration 31: Simplify $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$ if $\frac{a}{b}\tan x > -1$.   **(JEE MAIN)**

Sol: Divide the numerator and denominator inside the bracket by b cos x and expand.

We have, $\tan^{-1}\left[\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right] = \tan^{-1}\left[\dfrac{\dfrac{a\cos x - b\sin x}{b\cos x}}{\dfrac{b\cos x - a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\dfrac{\dfrac{a}{b} - \tan x}{1 + \dfrac{a}{b}\tan x}\right] = \tan^{-1}\dfrac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\dfrac{a}{b} - x$

Illustration 32: Solve the following equations:

(i) $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$ (ii) $2\tan^{-1}(\cos x) = \tan^{-1}(2\cosec x)$ **(JEE ADVANCED)**

Sol: Write $\dfrac{\pi}{4}$ as $\tan^{-1} 1$ and simplify.

(i) $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \tan^{-1}1 \Rightarrow \tan^{-1}\dfrac{x-1}{x-2} = \tan^{-1}1 - \tan^{-1}\dfrac{x+1}{x+2}$

$\Rightarrow \tan^{-1}\dfrac{x-1}{x-2} = \tan^{-1}\left(\dfrac{1 - \dfrac{x+1}{x+2}}{1 + \dfrac{x+1}{x+2}}\right) \Rightarrow \tan^{-1}\dfrac{x-1}{x-2} = \tan^{-1}\dfrac{x+2-x-1}{x+2+x+1}$

$\Rightarrow \tan^{-1}\dfrac{x-1}{x-2} = \tan^{-1}\dfrac{1}{2x+3} \Rightarrow \dfrac{x-1}{x-2} = \dfrac{1}{2x+3} \Rightarrow (2x+3)(x-1) = x-2$

$\Rightarrow 2x^2 + x - 3 = x - 2 \Rightarrow 2x^2 - 1 = 0 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$

(ii) $2\tan^{-1}(\cos x) = \tan^{-1}(2\cosec x)$

$\Rightarrow \tan^{-1}\left(\dfrac{2\cos x}{1 - \cos^2 x}\right) = \tan^{-1}(2\cosec x)$

$\Rightarrow \dfrac{2\cos x}{\sin^2 x} = 2\cosec x \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$

Illustration 33: Prove that: $\sin^{-1}\dfrac{3}{5} - \sin^{-1}\dfrac{8}{17} = \cos^{-1}\dfrac{84}{85}$ **(JEE MAIN)**

Sol: Covert the L.H.S. in terms of $\cos^{-1}$.

We have $\sin^{-1}\dfrac{3}{5} - \sin^{-1}\dfrac{8}{17} = \cos^{-1}\dfrac{4}{5} - \cos^{-1}\dfrac{15}{17}$ $\left[\because \sin^{-1}\dfrac{3}{5} = \cos^{-1}\dfrac{4}{5} \;\&\; \sin^{-1}\dfrac{8}{17} = \cos^{-1}\dfrac{15}{17}\right]$

$= \cos^{-1}\left\{\dfrac{4}{5}\times\dfrac{15}{17} + \sqrt{1 - \left(\dfrac{4}{5}\right)^2}\times\sqrt{1 - \left(\dfrac{15}{17}\right)^2}\right\} = \cos^{-1}\left\{\dfrac{4}{5}\times\dfrac{15}{17} + \dfrac{3}{5}\times\dfrac{8}{17}\right\} = \cos^{-1}\left\{\dfrac{60}{85} + \dfrac{24}{85}\right\} = \cos^{-1}\dfrac{84}{85}$

Illustration 34: Prove that: $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$ **(JEE MAIN)**

Sol: We have $\sin^{-1}\dfrac{4}{5} - \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \left\{\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13}\right\} + \sin^{-1}\dfrac{16}{65}$

$= \sin^{-1}\left\{\dfrac{4}{5}\sqrt{1-\left(\dfrac{5}{13}\right)^2} + \dfrac{5}{13}\sqrt{1-\left(\dfrac{4}{5}\right)^2}\right\} + \sin^{-1}\dfrac{16}{25}$

$= \sin^{-1}\left\{\dfrac{4}{5}\times\dfrac{12}{13} + \dfrac{5}{13}\times\dfrac{3}{5}\right\} + \sin^{-1}\dfrac{16}{25} = \sin^{-1}\dfrac{63}{65} + \sin^{-1}\dfrac{16}{25}$

$= \cos^{-1}\dfrac{16}{65} + \sin^{-1}\dfrac{16}{25}\left[\because \sin^{-1}\dfrac{63}{65} = \cos^{-1}\sqrt{1-\left(\dfrac{63}{65}\right)^2} = \cos^{-1}\dfrac{16}{65}\right]$

$= \dfrac{\pi}{2}\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

6. SIMPLIFICATION OF INVERSE FUNCTIONS BY ELEMENTARY SUBSTITUTION

(a) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ if $-1 \le x \le 1$

(b) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ if $-1 \le x \le 1$

(c) $2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & -1 \le x \le 1 \\[2mm] \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & 0 \le x \le 1 \\[2mm] \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & 0 \le x < \infty \end{cases}$

(d) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & x \ge 1 \\ -\pi - 2\tan^{-1}x & x \le -1 \end{cases}$

(e) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & x \ge 0 \\ -2\tan^{-1}x & x < 0 \end{cases}$

(f) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x & -1 < x < 1 \\ 2\tan^{-1}x - \pi & x > 1 \end{cases}$

(g) $\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \csc^{-1}\left(\dfrac{1}{x}\right)$

(h) $\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

(i) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

(j) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = \pi$, if $x \geq 1$

(k) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = -\pi$ if $x \leq -1$

(l) $\sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & -1 \leq x \leq 1/2 \\ 3\sin^{-1}x & -1/2 \leq x \leq 1/2 \\ \pi - 3\sin^{-1}x & 1/2 \leq x \leq 1 \end{cases}$

(m) $\cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & -1 \leq x \leq -1/2 \\ 2\pi - 3\cos^{-1}x & -1/2 \leq x \leq 1/2 \\ 3\cos^{-1}x & 1/2 \leq x \leq 1 \end{cases}$

(n) $\tan^{-1}\dfrac{3x - x^3}{1 - 3x^2} = \begin{cases} 3\tan^{-1}x & -\dfrac{1}{\sqrt{3}} < x\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & x > \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & x < -\dfrac{1}{\sqrt{3}} \end{cases}$

MASTERJEE CONCEPTS

While writing inverse trigonometric functions in their simplest forms, we use the following substitutions.

- For $\sqrt{a^2 - x^2}$, we substitute $x = a\sin\theta$ or $x = a\cos\theta$

- For $\sqrt{a^2 + x^2}$, we substitute $x = a\tan\theta$ or $x = a\cot\theta$

- For $\sqrt{x^2 - a^2}$, we substitute $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$

- For $\sqrt{a+x}$ and $\sqrt{a-x}$ occurring together or separately, we substitute $x = a\cos\theta$

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Illustration 35: Solve for x: $\sin\left(2\cos^{-1}\left(\cot(2\tan^{-1}x)\right)\right) = 0$ **(JEE ADVANCED)**

Sol: The R.H.S. is equal to zero implies $\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \dfrac{n\pi}{2}$ and proceed accordingly to find the value of x.

$$\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 0 \\ \dfrac{\pi}{2} & \text{if } n = 1 \\ \pi & \text{if } n = 2 \end{cases} \Rightarrow \cot\left(2\tan^{-1}x\right) = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$$\Rightarrow 2\tan^{-1}x = \begin{cases} n\pi + \dfrac{\pi}{4} \\ n\pi + \dfrac{\pi}{2} \\ n\pi - \dfrac{\pi}{4} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \dfrac{n\pi}{2} + \dfrac{\pi}{8} \\ \dfrac{n\pi}{2} + \dfrac{\pi}{4} \\ \dfrac{n\pi}{2} - \dfrac{\pi}{8} \end{cases} \Rightarrow \tan^{-1}x = \begin{cases} \dfrac{\pi}{8}, -\dfrac{3\pi}{8} \\ \dfrac{\pi}{4}, -\dfrac{\pi}{4} \\ -\dfrac{\pi}{8}, \dfrac{3\pi}{8} \end{cases}$$

$$\Rightarrow x = \pm 1, \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1)$$

Illustration 36: Solve the system of inequalities involving inverse circular functions arc $\tan^2 x - 3$ arc $\tan x + 2 > 0$ and $[\sin^{-1}x] > [\cos^{-1}x]$ where [ ] denotes the greatest integer function. **(JEE ADVANCED)**

Sol: Substitute $\tan^{-1} x$ equal to t.

$$\Rightarrow (t - 2)(t - 1) > 0$$

$$\Rightarrow t > 2 \text{ or } t > 1$$

$$\Rightarrow \tan^{-1} x > 2 \text{ or } \tan^{-1} x > 1$$

$x \in (-\infty, \tan 1) \quad x > \tan 1$

Again $[\sin^{-1}x] > [\cos^{-1}x]$

$\quad [\sin^{-1}x]$ can take the values $\{-2, -1, 0, 1\}$

And $[\cos^{-1}x]$ can take the values $\{0, 1, 2, 3\}$

Hence $[\sin^{-1}x]$ can be greater than $[\cos^{-1}x]$ only

If $[\sin^{-1}x] = 1$ and $[\cos^{-1}x] = 1$

Now, $[\sin^{-1}x] = 1 \Rightarrow 1 \le \sin^{-1}x \le \pi/2 \qquad (1 \le \sin^{-1} x < 2)$

$\quad \sin 1 \le x \le 1$

And $[\cos^{-1}x] = 0 \Rightarrow 0 \le \cos^{-1} x < 1$

$\quad \cos 1 < x \le 1$

Now, x must satisfy

From this $x \in [\sin 1, 1]$

PROBLEM-SOLVING TACTICS

- Making a habit of writing angle values in radians rather in degrees makes the calculation of inverse trigonometric functions easier.

- Try to remember graphs of inverse trigonometric functions. Sometimes it is easier to approximate answers using graphical methods.

- Always verify whether the results are in the range or domain of the respective function.

- In some cases, constructing a right angled triangle for the given inverse function and then solving using properties of triangle is much helpful.

- In case of identities in inverse circular functions, principal values should be taken. As such signs of x, y, etc., will determine the quadrant in which the angles will fall. In order to bring the angles of both sides in the same quadrant, one should make an adjustment by π.

FORMULAE SHEET

1.	If y = sin x, then x = sin⁻¹ y, similarly for other inverse T-functions.

2.	Domain and Range of Inverse T-functions:	
Function	**Domain(D)**	**Range (R)**
$\sin^{-1} x$	$-1 \leq x \leq 1$	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$
$\cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq \theta \leq \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$
$\cot^{-1} x$	$-\infty < x < \infty$	$0 < \theta < \pi$
$\sec^{-1} x$	$x \leq -1, x \geq 1$	$0 \leq \theta \leq \pi, \theta \neq \dfrac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	$x \leq -1, x \geq 1$	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}, \theta \neq 0$

3.	Properties of Inverse T-functions:
	(i) $\sin^{-1}(\sin \theta) = \theta$ provided $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ $\cos^{-1}(\cos \theta) = \theta$ provided $0 \leq \theta \leq \pi$ $\tan^{-1}(\tan \theta) = \theta$ provided $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ $\cot^{-1}(\cot \theta) = \theta$ provided $0 < \theta < \pi$

$\sec^{-1}(\sec\ \theta) = \theta$ provided $0 \leq \theta < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < \theta \leq \pi$

$\operatorname{cosec}^{-1}(\operatorname{cosec}\ \theta) = \theta$ provided $-\dfrac{\pi}{2} \leq \theta < 0$ or $0 < \theta \leq \dfrac{\pi}{2}$

(ii) $\sin\ (\sin^{-1} x) = x$ provided $-1 \leq x \leq 1$

$\cos\ (\cos^{-1} x) = x$ provided $-1 \leq x \leq 1$

$\tan\ (\tan^{-1} x) = x$ provided $-\infty < x < \infty$

$\cot\ (\cot^{-1} x) = x$ provided $-\infty < x < \infty$

$\sec\ (\sec^{-1} x) = x$ provided $-\infty < x \leq -1$ or $1 \leq x < \infty$

$\operatorname{cosec}\ (\operatorname{cosec}^{-1} x) = x$ provided $-\infty < x \leq -1$ or $1 \leq x < \infty$

(iii) $\sin^{-1}\ (-x) = -\sin^{-1} x,$

$\cos^{-1}\ (-x) = \pi - \cos^{-1} x$

$\tan^{-1}\ (-x) = -\tan^{-1} x$

$\cot^{-1}\ (-x) = \pi - \cot^{-1} x$

$\operatorname{cosec}^{-1}\ (-x) = -\operatorname{cosec}^{-1} x$

$\sec^{-1}\ (-x) = \pi - \sec^{-1} x$

(iv) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2},\quad \forall\ x \in [-1,\ 1]$

$\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2},\quad \forall\ x \in R$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2},\quad \forall\ x \in (-\infty, -1] \cup [1, \infty)$

4.	Value of one inverse function in terms of another inverse function:

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$= \sec^{-1}\dfrac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\dfrac{1}{x},\ 0 \leq x \leq 1$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} = \cot^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$= \sec^{-1}\dfrac{1}{x} = \operatorname{cosec}^{-1}\dfrac{1}{\sqrt{1-x^2}},\ 0 \leq x \leq 1$

(iii) $\tan^{-1} x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \cot^{-1}\dfrac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\dfrac{\sqrt{1+x^2}}{x},\ x \geq 0$

(iv) $\sin^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1} x,\ \forall\ x \in (-\infty, 1] \cup [1, \infty)$

(v) $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1} x,\ \forall\ x \in (-\infty, 1] \cup [1, \infty)$

(vi) $\tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{for}\ x > 0 \\ -\pi + \cot^{-1} x & \text{for}\ x < 0 \end{cases}$

5.	Formulae for sum and difference of inverse trigonometric function:

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$; if $x > 0,\ y > 0,\ xy < 1$

(ii) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$; if $x>0,\ y>0,\ xy > 1$

(iii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1+xy}\right)$; if $xy > -1$

(iv) $\tan^{-1} x - \tan^{-1} y = \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$; if $x>0,\ y<0,\ xy < -1$

(v) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(vi) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right]$;

If $x, y, \geq 0$ & $x^2 + y^2 \leq 1$

(vii) $\sin^{-1} x \pm \sin^{-1} y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right]$;

If $x, y, \geq 0$ & $x^2 + y^2 > 1$

(viii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\ \sqrt{1-y^2} \right]$;

If $x, y, > 0$ & $x^2 + y^2 \leq 1$

(ix) $\cos^{-1} x \pm \cos^{-1} y = \pi - \cos^{-1}\left[xy \mp \sqrt{1-x^2}\ \sqrt{1-y^2} \right]$;

If $x, y, > 0$ & $x^2 + y^2 > 1$

6.	Inverse trigonometric ratios of multiple angles

(i) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$, if $-1 \leq x \leq 1$

(ii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$, if $-1 \leq x \leq 1$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(iv) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(v) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(vi) $3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

3 MATRICES

1. **Matrices** - A system of mn numbers (real or complex) arranged in a rectangular array of m rows and n columns is called a matrix of order m × n. An m × n matrix (to be read as 'm by n' matrix)
An m × n matrix is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The numbers a_{11}, a_{12} etc are called the elements or entries of the matrix. If A is a matrix of order m × n, then we shall write $A = [a_{ij}]_{m \times n}$ where, a_{ij} represent the number in the i-th row and j-th column.

2. **Row Matrix** - A single row matrix is called a row matrix or a row vector. e.g. the matrix $[a_{11}, a_{12}, \ldots a_{1n}]$ is a row matrix.

3. **Column Matrix** - A single column matrix is called a column matrix or a column vector. e.g. the matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ a_{m1} \end{bmatrix}$ is a m × 1 column matrix.

4. **Order of a Matrix** - A matrix having m rows and n columns is of the order m × n. i.e. consisting of m rows and n columns is denoted by $A = [a_{ij}]_{m \times n}$.

5. **Square Matrix** - If m = n, i.e. if the number of rows and columns of a matrix are equal, say n, then it is called a square matrix of order n.

6. **Null or Zero Matrix** - If all the elements of a matrix are equal to zero, then it is called a null matrix and is denoted by $O_{m \times n}$ or 0.

7. **Diagonal Matrix** - A square matrix, in which all its elements are zero except those in the leading diagonal is called a diagonal matrix, thus in a diagonal matrix, $a_{ij} = 0$, if $i \neq j$, e.g. the diagonal matrices of order 2 and 3 are $\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, $\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}$

8. **Scalar Matrix** - A square matrix in which all the diagonal element are equal and all other elements equal to zero is called a scalar matrix.
i.e. in a scalar matrix $a_{ij} = k$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$. Thus $\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$ is a scalar matrix.

9. **Unit Matrix or Identity Matrix** - A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or identity matrix.

    e.g. a unit or identity matrix of order 2 and 3 are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ respectively.

10. **Upper triangular Matrix** - A square matrix A whose elements $a_{ij} = 0$ for i > j is called an upper triangular matrix.

11. **Lower triangular Matrix** - A square matrix A whose elements $a_{ij} = 0$ for i < j is called a lower triangular matrix.

12. **Equal Matrices** - Two matrices A and B are said to be equal, written as A = B if
    (i) they are of the same order i.e. have the same number of rows and columns, and
    (ii) the elements in the corresponding places of the two matrices are the same.

13. **Transpose of a matrix** - Let A be a m × n matrix then the matrix of order n × m obtained by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A′ or A^T.

14. **Negative of Matrix** - Let A = $[a_{ij}]_{m \times n}$ be a matrix. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by –A.

15. **Symmetric Matrix** - a square matrix A is said to be symmetric if A′ = A
    Thus a square matrix A = $[a_{ij}]$ is symmetric if A = $[a_{ij}]$ is symmetric if $a_{ij} = -a_{ji}$ for all values of i and j.

16. **Skew-Symmetric Matrix** - A square matrix A is said to be skew-symmetric if A′ = –A Thus a square matrix A = $[a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all values of i and j.
    In particular $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$ i.e. all diagonal elements of a skew-symmetric matrix are o.

17. For any square matrix A with real number entries, A + A′ is a symetric matrix and A – A′ is a skew symetric matrix.

18. Any square matrix can be expressed as the sum of a symetric and a skew symetric matrix.
    If A be a square matrix, then we can write $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, here $\frac{1}{2}(A + A')$ is symetric matrix and $\frac{1}{2}(A \quad A)$ is skew symetric matrix.

19. **Addition of Matrices** - Let there be two matrices A and B of the same order m × n. then the sum denoted by A + B is defined to be the matrix of order m × n obtained by adding the corresponding elements of A and B.
    Thus if A = $[a_{ij}]_{m \times n}$ and B = $[b_{ij}]_{m \times n}$ then A + B = $[a_{ij} + b_{ij}]_{m \times n}$

20. **Scalar Multiplication of a Matrix** - Let A = $[a_{ij}]_{m \times n}$ be a matrix and K is a scalar. Then the matrix obtained by multiplying each element of matrix A by K is called the scalar multiplication of matrix A by K and is denoted by KA or AK.

21. **Multiplication of Matrices** - Product of two matrices exists only if number of column of first matrix is equal to the number of rows of the second. Let A be m × n and B be n × p matrices. Then the product of matrices A and B denated by A.B is the matrix of order m × p whose (i, j)th element is obtained by adding the products of corresponding elements of ith row of A and jth column of B.

22. **Elementary Row Operations** - The operations known as elementary row operations on a matrix are-
    (i) The interchange of any two rows of a matrix. (The notations $R_i \leftrightarrow R_j$ is used for the interchange of the i-th and j-th rows.)
    (ii) The multiplication of every element of a row by a non-zero element (constant).
        (The notations $K.R_i$ is used for the multiplication of every element of i-th row by a constant K.
    (iii) The addition of the elements of a row, the product of the corresponding elements of any other row by any non-zero constant. (The notation $R_i + K.R_j$ is generally used for addition to the elements of i-th row to the element of j-th row multiplied by the constant K (K ≠ 0))

23. **Invertible matrices** - If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the Inverse matrix of A and it is denoted by A^{-1}. In that care A is said to be invertible.

24. If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}. A^{-1}$.

25. **Inverse of a matrix by elementry operations** - Let X, A and B be matrices of, the same order such that X = AB. In order to apply a sequence of elementry row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Similarly, in order to apply a sequence of elementry column operations on the matrix equation X = AB, we will apply, these operations simultaneously on X and on the second matrix B of the product AB on RHS. In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementry row operations, write A = IA and apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we with to find A^{-1} using column operations, then, write A = AI on A = IA till we get, I = BA. The matrix and apply a sequence of column operations on A = AI till we get, I = AB.

Remark - In case, after applying one or more elementry row (column) operations on A = IA (A = AI). If we obtain all zero in one or more rows of the matrix A on L.H.S., that A^{-1} does not exist.

CONNECTING CONCEPTS

1. The elements a_{ij} of a matrix for which i = j are called the diagonal elements of a matrix and the line along which all these elements lie is called the principal diagonal or the diagonal of the matrix.

2. **Properties of transpose of the matrices-**
   (i) $(A + B)' = A' + B'$
   (ii) $(KA)' = KA'$, where K is constant
   (iii) $(AB)' = B'A'$
   (iv) $(A')' = A$

3. **Properties of Matrix addition-**
   (i) **Matrix Addition is Commutative** - If A and B be two m × n matrices, then A + B = B + A
   (ii) **Matrix Addition is Associative** - If A, B and C be three m × n matrices, then
   $$(A + B) + C = A + (B + C)$$

4. **Properties of Multiplication of a Matrix by a Scalar-**
   (i) If K_1 and K_2 are scalars and A be a matrix, then $(K_1 + K_2) A = K_1 A + K_2 A$.
   (ii) If K_1 and K_2 are scalars and A be a matrix, then $K_1 (K_2 A) = (K_1 K_2) A$.
   (iii) If A and B are two matrices of the same order and K, a scalar, then $K (A + B) = KA + KB$.
   (iv) If K_1 and K_2 are two scalars and A is any matrix then $(K_1 + K_2) A = K_1 A + K_2 A$.
   (v) If A is any matrix and K be a scalar.
   then $(-K) A = -(KA) = K (-A)$.

5. **Properties of Matrix Multiplication -**
   (i) **Associative law for Multiplication** - If A, B and C be three matrices of order m × n and n × p and p × q, respectively, then (AB) C = A (BC).
   (ii) **Distributive Law** - If A, B, C be three matrices of order m × n, n × p and n × q respectively.
   then $A \cdot (B + C) = A \cdot B + A \cdot C$
   (iii) Matrix Multiplication is not commutative.
   i.e. $A \cdot B \neq B \cdot A$
   (iv) **The existence of multiplicative Identity :** For every square matrix A, there exists an identity matrix of same order such that IA = AI = A.

6. If A be any n × n square matrix, then
   $$A \cdot (Adj A) = (Adj A) \cdot A = |A|. I_n$$
   where I_n is an n × n unit matrix

7. (i) Only square matrix can have inverse
   (ii) The matrix $B = A^{-1}$, will also be a square matrix of same order A.
   (iii) The square matrix A is said to be invertible if A^{-1} exists.

8. Every invertible matrix possesses a unique inverse.

4 DETERMINANTS

1. **Determinant -** (i) A determinant is a particular type of expression written in a special concise form of rows and columns, equal in number.

   For example $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant having 2 rows and 2 columns, hence it is of second order. The numbers a_1, b_1, a_2, b_2 are called the elements of the determinant. The value of the above determinant of

   third order is written as $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. It has three rows and three columns.

   The number of elements $= 3^2 = 9$. In general, the number of elements in a determinant of order $n = n^2$.

   (ii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix then determinant of matrix A is written as $|A|$ or det $(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

   (iii) Only square matrices have determinants.

2. **Minors -** The determinant obtained by deleting the i-the row and j-th column passing through the element a_{ij} is called the minor of element a_{ij} and is denoted by M_{ij}.

3. **Cofactors -** The cofactor of element a_{ij} is $(-1)^{1+j}$ times the determinant obtained by deleting the i-th row and jth column passed through a_{ij} and is denoted by C_{ij} i.e. $C_{ij} = (-1)^{1+j} M_{ij}$

4. **Values of the determinant -** The sum of the products of elements of any row (column) by the corresponding co-factors is equal to the value of the determinant.

   let $\Delta = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, Then $\Delta = a_{11} c_{11} + a_{12} c_{12} + a_{13} c_{13}$

5. **Area of a Triangle -** The area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

   $= \dfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

   (i) The area is positive, take only absolute value.

   (ii) If the three points are collinear, the area of triangle is zero.

6. $|AB| = |A| \, |B|$

7. A square matrix is invertible if and only if A is non-singular.

8. **Linear system of Equations -**

   Consistent System - The system of equation is said to be consistent if it has one or more then one solutions.

Inconsistent System - The system of equation is inconsistent if it has no solution

Consider the system of equation

$$a_1x + b_1y_+ c_1z = d_1$$
$$a_2x + b_2y_+ c_2z = d_2$$
$$a_3x + b_3y_+ c_3z = d_3$$

let
$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The given system of equation can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $$AX = B$$
$$\therefore \qquad X = A^{-1} B.$$

9. **Consistence/Inconsistence of system of Equations**

(a) **For a non-homogeneous system of equation AX ≠ 0**

  (i) If $|A| \neq 0$, AX = B has a unique solution.

  (ii) If $|A| = 0$, and (adj A) B ≠ 0
  then the system of equation is inconsistent.

  (iii) If $|A| = 0$ and (adj A) B = 0, then the system of equation has infinitely many solutions.

(b) **For the homogeneous system of equation AX = 0**

  (i) If $|A| \neq 0$, the solution is x = 0, y = 0, z = 0. This is called the trivial solution.

  (ii) If $|A| = 0$, the system has infinitely many solution. In such a case, we put one of the variables equal to k. let z = k, then we find the value of x and y in terms of k.

10. **Adjoint of a Determinant** - The adjoint of a square matrix is the transpose of matrix cofactors. If A_{ij} is the cofactor of a_{ij} of det A or $|a_{ij}|$, the

$$\text{adj A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

11. **Inverse of a matrix** - Inverse of a matrix A, $A^{-1} = \dfrac{1}{|A|}$ adj A ; if $|A| \neq 0$ i.e., matrix A is invertible or non-singular.

12. If A is a square matrix, then A (adj A) = (adj A) A = $|A|$. I

13.   (i) $(AB)^{-1} = B^{-1} . A^{-1}$ (ii) $A^{-1} = (A^{-1})^T$ (iii) $(A^{-1})^{-1} = A$

CONNECTING CONCEPTS

1. The value of the determinant does not change when rows and columns are interchanged. The determinant obtained by interchanging the rows and columns is called the transpose of the determinant and is denoted by Δ^T. Thus $\Delta = \Delta^T$.

2. If all the elements of a row (column) are zero, then the value of the determinant is zero.

3. The interchange of any two rows of the determinant changes its sign.
Thus if Δ^* is the new determinant obtained on interchanging any two rows (columns), then
$$\Delta = -\Delta^*$$

If i-th and j-th row are interchanged then this operation is denoted by $R_i \longleftrightarrow R_j$.

4. If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant. Thus if we apply $R_i \to pR_i$, i.e, each element of i-th row is multiplied by p, then we get

$$\Delta^* = p\Delta \quad \text{or} \quad \Delta = \frac{1}{p}\,\Delta^*\ (P \ne 0)$$

5. If all the elements of a row (column) are proportional (identical) to the elements of some other row (column) then determinant is zero.

6. If each element of any row (column) is sum of two numbers, the determinant can be expressed as the sum of two determinants of the same order eg.

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

7. The value of a determinant remains unaltered under an operation of the form

$$R_i \to R_i + pR_j \to \text{similarly, for columns i.e., operation of the form } C_i \to C_i + pC_j + q\,C_k;\, j,\, k \ne i$$

8. If a determinant $\Delta\ (x)$ becomes zero on putting $x = \alpha$, then $(x - \alpha)$ is a factor of $\Delta\ (x)$.

9. Determinant which have all elements equal to zero except the diagonal elements, is equal to the product of the diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

5 CONTINUITY AND DIFFERENTIABILITY

1. **Continuity** - A real valued function $f(x)$ of variable x defined on an interval I is said to be continuous at $x = a \in I$, $\lim_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$.

   $\therefore$ $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$, where 'h' is a very small +ve quantity.

2. A function $f(x)$ is said to be continuous in an interval I, if it is continuous at each point of the interval.

3. **Discontinuity** - A function said to be discontinuous at a point $x = a$, if it is not continuous at this point. This point $x = a$ where the function is not continuous is called the point of discontinuity.

4. Suppose f and g be two real functions continuous at a real number c, then
   (i) $f + g$ is continuous at $x = c$
   (ii) $f - g$ is continuous at $x = c$
   (iii) $f \cdot g$ is continuous at $x = c$
   (iv) $\dfrac{f}{g}$ is continuous at $x = c$, (provided $g(c) \neq 0$)

5. (i) If g is a continuous function, then $\dfrac{1}{g}$ is also continuous.

   (ii) Suppose f and g are real valued functions such that (fog) is defined at c. If f and g is continuous at c then (fog) is also continuous at c.

6. **Differentiability** - The concept of differentiability has been introduced in the lower class let f be a real function and c is a point in its domain. The derivative $f'(c)$ of f at c is defined as $\lim_{h \to 0} \dfrac{f(c+h)-f(c)}{h}$, provided limit exists

   Thus, $f'(c) = \dfrac{d}{dx} [f(x)]_c$. $f'(x)$ is defined as $f'(x) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

   Every differentiable function is continuous.

7. **Algebra of Derivatives** - Let u, v be the function of x.
   (i) $(u \pm v)' = u' \pm v'$
   (ii) $(uv)' = u'v + uv'$
   (iii) $\left(\dfrac{u}{v} \right)' = \dfrac{u'v - uv'}{v^2}$, where $v \neq 0$.

8. **Chain Rule** - If f and g are differentiable functions in their domain, then fog (x) or f g (x) is also differentiable and $(fog)'(x) = f' g(x) \times g'(x)$

   More easily if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

   If y is a function of u, u is a function of v and v is a function of x then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dv} \times \dfrac{dv}{dx}$.

9. **Implicit functions** - An equation in the form $f(x, y) = 0$ in which y is not expressible in terms of x is called as an implicit function of x and y.

Both sides of equations are differentiated term wise with respect to x then from this equation $\dfrac{dy}{dx}$ is obtained. It may be noted that when a function of y occurs, then differentiate it w.r.t. y and multiply it by $\dfrac{dy}{dx}$.

Collect the terms containing $\dfrac{dy}{dx}$ at one side and find $\dfrac{dy}{dx}$

10. **Exponential function** - The exponential function with positive base b > 1, is the function $y = b^x$.
   (i) The graph of $y = 10^x$ is
   (ii) Domain = R
   (iii) Range = R^+
   (iv) The point (0, 1) always lies on the graph.
   (v) It is an increasing function
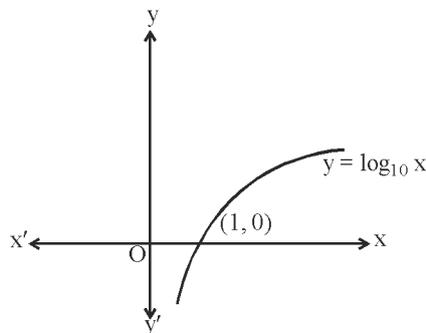   (vi) As $x \to -\infty \ y \to 0$
   (vii) $\dfrac{d}{dx} a^x = a^x \log_e a, \ \dfrac{d}{dx} e^x = e^x.$
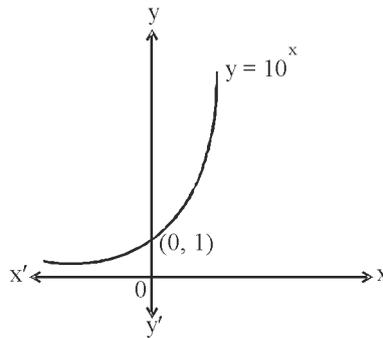
11. **Logarithmic function** - Let b > 1 be a real number. $b^x = a$ may be written as $\log_b a = x$.
   (i) The graph of $y = \log_{10} x$ is
   (ii) Domain = R^+
   (iii) Range = R
   (iv) It is an increasing function.
   (v) As $x \to 0, y \to -\infty$.
   (vi) The function $y = e^x$ and $y = \log_e x$ are the mirror images of each other
   (vii) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}\log_a e, \ \dfrac{d}{dx}\log_e x = \dfrac{1}{x}$

12. **Derivatives of functions in Parametric form** - The set of equations x = f (t), y = g (t) is called the parametric form of an equation.

Now, $\dfrac{dx}{dt} = f'(t), \dfrac{dy}{dt} = g'(t)$, $\therefore \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ or $\dfrac{g'(t)}{f'(t)}$

13. **Second order derivative-** let $y = f(x)$ then $\dfrac{dy}{dx} = f'(x)$

If $f'(x)$ is differentiable, then it is again differentiated and get

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ or $\dfrac{d^2y}{dx^2} = f''(x)$

$\dfrac{d^2y}{dx^2}$ or $f''(x)$ is called the second derivative of y or $f(x)$ with respect to x.

14. **Rolle's Theorem -** Let $f : [a, b] \to R$ be continuous an closed interval $[a, b]$ and differentiable an open interval (a, b) such that $f(a) = f(b)$ where a, b are real numbers, then there must exists at least one value $c \in (a, b)$ of x,such that $f'(c) = 0$.
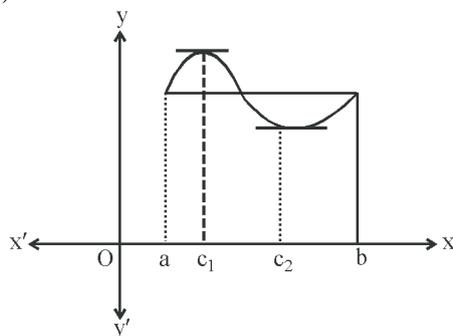
We observe that $f(a) = f(b)$, There exists two point c_1 and $c_2 \in (a, b)$ such that $f'(c_1) = 0$ and $f'(c_2) = 0$, i.e. Tangent at c_1 and c_2 are parallel to x-axis.

15. **Mean Value Theorem-** Let $f : [a, b] \to R$ be a continuous function on the closed interval $[a, b]$ and differentiable in the open interval (a, b), then there must exists at least one value $c \in (a, b)$ of x, such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
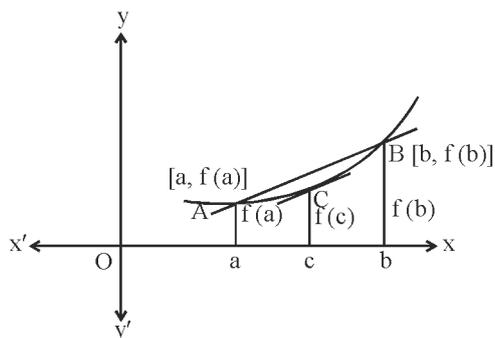
Here, $\dfrac{f(b) - f(a)}{b - a}$ is the slope of secant drawn between A $[a, f(a)]$ and B $[b, f(b)]$. There is at least one point $c \in (a, b)$ of x where slope of the tangent at $x = c$ is parallel to chord AB.

CONNECTING CONCEPTS

Some common type functions as constant function, Identity function, implicit function, Modulus function, Exponential function, and logarithmic function are continuous in their domains.

1. Every polynomial function is differentiable at each $x \in R$.
2. The exponential function a^x, $a > 0$, is differentiable at each $x \in R$

3. Every constant function is differentiable at each $x \in R$.

4. The logarithmic function is differentiable at each point in its demain.

5. Trigonometric and inverse-trigenometric functions are differentiable in their domains.

6. The sum, difference, product and quotient of two differentiable functions is differentiable

7. The composition of differentiable function is differentiable function.

8. (i) $\log_b pq = \log_b p + \log_b q$

  (ii) $\log_b \dfrac{p}{q} = \log_b p - \log_b q$

  (iii) $\log_b p^x = x \log_b p$

  (iv) $\log_a p = \dfrac{\log_b p}{\log_b a}$

9. Derivativs of Inverse Trigonometric Functions.

| Functions	Domain	Derivative
$\sin^{-1} x$	$[-1, 1]$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\cos^{-1} x$	$[-1, 1]$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\tan^{-1} x$	R	$\dfrac{1}{1 + x^2}$
$\cot^{-1} x$	R	$\dfrac{-1}{1 + x^2}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\dfrac{1}{x\sqrt{x^2 - 1}}$
$\mathrm{cosec}^{-1} x$	$(-\infty, -1) \cup [1, \infty)$	$\dfrac{-1}{x\sqrt{x^2 - 1}}$

6 APPLICATION OF DERIVATIVES

KEY CONCEPTS INVOLVED

1. **Rate of change of Quantities** – Let $y = f(x)$ be a function. If the change in one quantity y varies with another quantity x, then $\dfrac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y with respect to x. $\dfrac{dy}{dx}\bigg]_{x = x_0}$ or $f'(x_0)$ represents the rate of change of y w.r.t. x at $x = x_0$.

2. **Increasing and Decreasing function at x_0** A function f is said to be

   (a) Increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$
   Alternatively, if $f'(x) \geq 0$ for each x in (a, b)
   (b) Decreasing on (a, b) if $x_1 < x_2$ in (a, b) $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a, b)$ Alternatively, if $f'(x) \leq 0$ for each x in (a, b)

3. **Test : Increasing/decreasing/constant function** – Let f be a continuous on [a, b] and differentiable in an open interval (a, b), then.
   (i) f is increasing on [a, b], if $f'(x) > 0$ for each $x \in (a, b)$
   (ii) f is decreasing on [a, b], if $f'(x) < 0$ for each $x \in (a, b)$
   (iii) f is constant on [a, b], if $f'(x) = 0$ for each $x \in (a, b)$

4. **Tangent to a Curve** – Let $y = f(x)$ be the equation of a curve. The equation of the tangent at (x_0, y_0) is
   $y - y_0 = m(x - x_0)$, where m = slope of the tangent $= \dfrac{dy}{dx}\bigg]_{(x_0, y_0)}$ or $f'(x_0)$

5. **Normal to the Curve** – Let $y = f(x)$ be the equation of the curve Equation of the normal at (x_0, y_0) is
   $$y - y_0 = -\dfrac{1}{m}(x - x_0)$$
   where $\qquad m$ = Slope of the tangent at (x_0, y_0)
   $$= \dfrac{dy}{dx}\bigg]_{(x_0, y_0)} \text{ or } f'(x_0)$$

6. **Approximation** – Let $y = f(x)$, Δx be a small increament in x and Δy be the increament in y corrseponding to the increament in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of $\Delta y = \left(\dfrac{dy}{dx}\right)\Delta x$

7. **Maximum Value, Minimum value, Extreme Value** – Let f be a function defined in the interval I, then
   (i) **Maximum Value** – If there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$ then $f(c)$ is called maximum value of f in I. The point c is known as a point of maximum value of f in I.
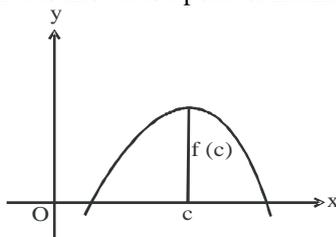
(ii) **Minimum Value** – If three exists a point c in I such that $f(c) \le f(x)$, $\forall x \in I$, then $f(x)$ is called the minimum value of f in I. The point c is called as a point of minimum value of f in I
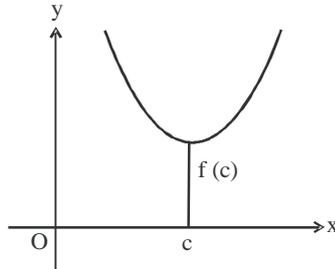
(iii) **Extreme Value** – If there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I, then $f(c)$ is the extreme value of $f(x)$ in I.
The point c is said to be an extreme point.
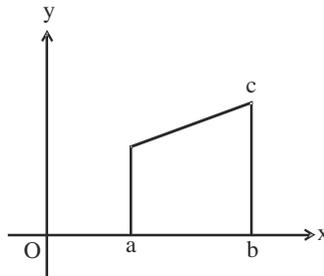
8. **Absolute Maxima and Minima** – let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Similarly, f has the absolute minimum value and attains at least once in I
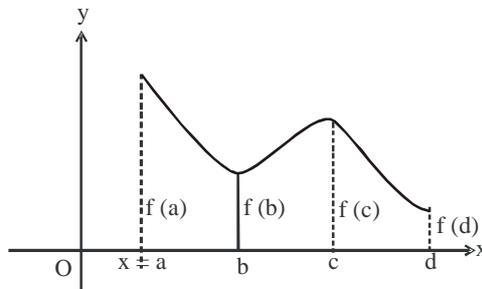
At x = b, there is a local minima
At x = c, there is a local maxima
At x = a, f (a) is the greatest value or absolute max. value.
At x = d, f (d) is the least value or absolute min. value.

9. **Local Maxima and Minima** – let f be a real valued function and c be an interior point in the domain of f, then
 (a) **Local Maxima** – c is a point of local maxima if there is an h > 0, such that $f(c) \ge f(x)$ for all $x \in [c-h, c+h)$
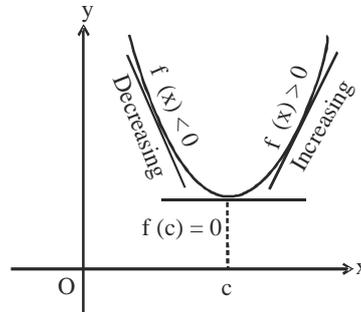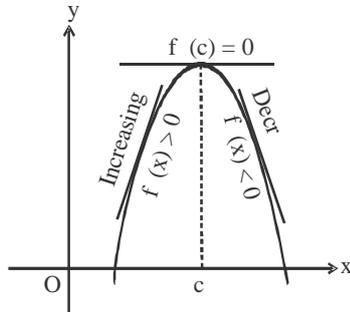The value $f(c)$ is called local maximum value of f.
 (b) **Local Minima** – c is a point of local minima if there is an h > 0, such that $f(c) \le f(x)$ for all $x \in (c-h \, c+h)$
The value of $f(c)$ is known as the local minimum value of f.
Geometrically – If x = c is a point of local maxima of f, then

f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$
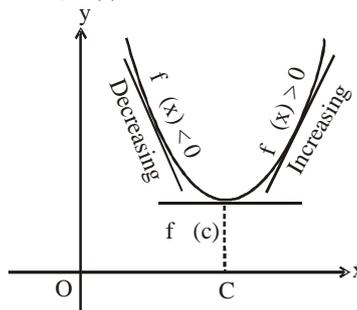
$\Rightarrow \qquad\qquad f'(c) = 0$

Similarly, if $x = c$ is a point of local minima of f, then f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$.

$\Rightarrow \qquad\qquad f'(c) = 0$

10. **Test of Local Maxima and Minima –**
   (i) Let f be a differentiable function defined on an open interval I and $c \in I$ be any point. f has a local maxima or a local minima at $x = c$, $f'(c) = 0$

   (ii) If $f'(x)$ changes sign from positive to negative as x increases from left to right through c i.e.,
      (a) $f'(x) > 0$ at every point in $(c - h, c)$
      (b) $f'(x) < 0$ at every point in $(c, c + h)$
      Then c is called a point of local maxima of f and $f(c)$ is local maximum value of f.
   (iii) If $f'(x)$ changes sign from negative to positive as x increase from left to right through c i.e.,
      (a) $f'(x) < 0$ at every point in $(c - h, c)$
      (b) $f'(x) > 0$ at every point in $(c, c + h)$
      Then c is called a point of local minima of f and $f(c)$ is a local minimum value of f.
   (iv) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called point of inflection.

11. **Second Derivative Test of Local Maxima and Minima –** let f be a twice differentiable function defined on an interval I and $c \in I$ and f be differentiable at $c \in I$, then,
   (i) $x = c$ is a local maxima,
      if $f'(c) = 0$ and $f''(c) < 0$.
      $f(c)$ is the local maximum value of f
   (ii) $x = c$ is a local minima, if $f'(c) = 0$ and $f''(c) > 0$
      $f(c)$ is the local minimum value of f.
   (iii) Point of inflection If $f'(c) = 0$ and $f''(c) = 0$
      Test fails. Then we apply first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

12. To find absolute maximum value or absolute minimum value –
  (i) Find all the critical points where $f'(x) = 0$
  (ii) Consider the end point also.
  (iii) Calculate the functional values at all the points found in step (i) and (ii)
  (iv) Identify the maximum and minimum values out of the values calculated in step (iii). These are absolute maximum and absolute minimum values.

CONNECTING CONCEPTS

1. **Increasing Function** – f is said to be increasing on I, if $x_1 < x_2$ on I, then $f(x_1) \leq f(x_2)$. for all $x_1, x_2 \in I$.
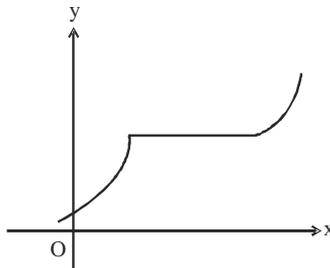
2. **Strictly Increasing function** – f is said to be strictly increasing on I, if $x_1 < x_2$ in I then $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.
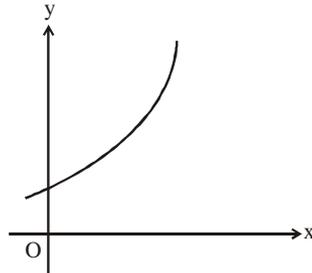
3. **Decreasing function** – f is said to be decreasing function on I, if $x_1 < x_2$ in I, then $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.
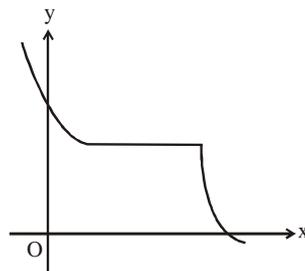
4. **Strictly Decreasing function** – f is said to be strictly decreasing function on I, if $x_1 > x_2$ in I then $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.
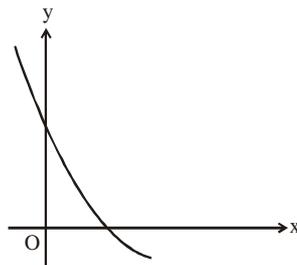
5. **Particular case of tangent** – Let $m = \tan \theta$

If $\theta = 0, m = 0$

Equation of tangent is $y - y_0 = 0$ i.e., $y = y_0$

If $\qquad \theta = \dfrac{\pi}{2}$, m is not defined.

$\therefore \qquad (x - x_0) = \dfrac{1}{m}(y - y_0)$

when $\qquad \theta = \dfrac{\pi}{2}$, $\cot \dfrac{\pi}{2} = 0$

$\therefore$ Equation of tangent is $x - x_0 = 0$ or $x = x_0$

7	INTEGRALS

1. **Integration** – The process of finding the function $f(x)$ whose differential coeffiicient w.r.t. 'x', denoted by

   $F(x)$ is given, is called the integration of $f(x)$ w.r.t. x and is written as $\int F(x)\,dx = f(x)$

   Thus, integration is an inverse process of differentiation or integration is anti of differentiation.

   The differential coefficient of a constant is zero. Thus if c is an arbitrary constant independent of x. then

   $\dfrac{d}{dx}[f(x)+c] = F(x)$ Thus $\int F(x)\,dx = f(x)+c$

   The arbitrary constant c is called the constant of integration.

2. **Integration by Substitution**

   (a) **To evaluate the integral** $\int f(ax+b)\,dx$

   Put $ax + b = t$, so that $adx = dt$ *i.e.,* $dx = \dfrac{1}{a}dt$

   $\int f(ax+b)\,dx = \int f(t)\cdot\dfrac{1}{a}dt = \dfrac{1}{a}F(t),$ where $\int f(t)\,dt = F(t) = F(ax+b)$

   If a function is not in some suitable form to find the integration, then we transform it into some suitable form by changing the independent variable x to t by substituting $x = g(t)$.

   Consider $\qquad I = \int f(x)\,dx$

   Put $\qquad\qquad x = g(t),$ so that $\dfrac{dx}{dt} = g'(t)$

   We write $\qquad dx = g'(t)\,dt$

   Thus $\qquad\qquad I = \int f(x)\cdot dx = \int f(g(t))\,g'(t)\,dt$

   But it is very important to guess, what will be the useful substitution.

   (b)(b) $\displaystyle\int \dfrac{f'(x)}{f(x)}\,dx = \log f(x) + c$

   (c) $\displaystyle\int [f(x)]^n f'(x)\,dx = f(x)^{n+1}\big/(n+1) + c$

   (d) Some important substitutions

function	Substitutions
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

3. **Trigonometrical transformations** – For the integration of the trigonometrical products such as $\sin^2 x,\ \cos^2 x,\ \sin^3 x,\ \cos^3 x,\ \sin ax\cos bx$ etc. they are expressed as the sum or difference of the sines and cosines of multiples of angles.

4. Integration of Some Special Integrals –

(a) For $\int \dfrac{dx}{ax^2 + bx + c}$, $\int \dfrac{dx}{\sqrt{ax^2 + bx + c}}$ and $\int \sqrt{ax^2 + bx + c}\, dx$

$$ax^2 + bx + c = a\left[x^2 + \dfrac{b}{a}x + \dfrac{c}{a} \right] = a\left[\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{c}{a} - \dfrac{b^2}{4a^2} \right] = a\left[\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a^2} \right]$$

Put $x + \dfrac{b}{2a} = t$, $\therefore$ $dx = dt$, $\dfrac{4ac - b^2}{4a^2} = \pm k^2$, $ax^2 + bx + c$ changes to $t^2 + k^2$, $t^2 - k^2$ or $k^2 - t^2$

(b) For $\int \dfrac{(px + q)\, dx}{ax^2 + bx + c}$, $\int \dfrac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$, $\int (px + q) \sqrt{(ax^2 + bx + c)}\, dx$

Put $px + q = A \dfrac{d}{dx}(ax^2 + bx + c) + B$

Compare the two sides and find the value of A and B.

Thus $\int \dfrac{px + q}{ax^2 + bx + c}dx = \int \dfrac{A \dfrac{d}{dx}(ax^2 + bx + c) + B}{(ax^2 + bx + c)}$

$$= A\int \dfrac{\dfrac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)}dx + B \int \dfrac{dx}{(ax^2 + bx + c)}$$

Similarly $\int \dfrac{px + q}{\sqrt{ax^2 + bx + c}}dx = A \int \dfrac{\dfrac{d}{dx}(ax^2 + bx + c)}{\sqrt{ax^2 + bx + c}}dx + B \int \dfrac{dx}{\sqrt{ax^2 + bx + c}}$

same as do $\int (px + q) \sqrt{ax^2 + bx + c}\, dx$.

(c) For $\int \dfrac{dx}{(x + k) \sqrt{ax^2 + bx + c}}$ put $x + k = \dfrac{1}{t}$

(d) For $\int \dfrac{dx}{\sqrt{(x - \alpha)(x - \beta)}}$, $\int \sqrt{\dfrac{x - \alpha}{\beta - x}}\, dx$

$\int \sqrt{(x - \alpha)(x - \beta)}\, dx$, Put $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

(e) For $\int \dfrac{dx}{a + b \cos x}$, $\int \dfrac{dx}{a + b \sin x}$, $\int \dfrac{dx}{a + b \cos x + c \sin x}$

$\sin x = \left(2 \tan \dfrac{x}{2} \right) / \left(1 + \tan^2 \dfrac{x}{2} \right)$, $\cos x = \left(1 - \tan \dfrac{x}{2} \right) / \left(1 + \tan^2 \dfrac{x}{2} \right)$ then put $\tan x/2 = t$

(f) For $\int \dfrac{p \cos x + q \sin x}{a + b \cos x + b \sin x}\, dx$

Put $p \cos x + q \sin x = A (a + b \cos x + b \sin x) + B$ differential of $(a + b \cos x + b \sin x) + C$

A, B and C can be calculated by equating the coefficients of cos x. sin x and the constant terms.

5. Integration by parts $\int u \cdot v \, dx = u \cdot \int v \, dx - \int \left[\dfrac{du}{dx} \cdot \int v\, dx \right] dx$

i.e., the integral of the product of two functions = (first function) × (Integral of the second function – Integral of {(dfferential of first function) x (Integral of second function)}

This formula is called integration by parts.

6. Partial Integration – To Evaluate $\int \dfrac{P(x)}{Q(x)} \, dx$

The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors.

If $\dfrac{P(x)}{Q(x)}$ is improper fraction, i.e., degree of numerator is equal or greater than the degree of denominator. Then first we reduce in proper rational function as $\dfrac{P(x)}{Q(x)} = T(x) + \dfrac{P_1(x)}{Q(x)}$ where $T(x)$ is a polynomial in x

and $\dfrac{P_1(x)}{Q(x)}$ is a proper rational function.

After this, the integration can be carried out easily using the already known methods. The following Table 7.1 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.1

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px - q}{(x-a)(x-b)}, \ a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$
2.	$\dfrac{px + q}{(x-a)^2}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-b)^2}$
3.	$\dfrac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
4.	$\dfrac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$
5.	$\dfrac{px^2 + qx + r}{(x-a)(x^2 + bx + c)}$ Where $x^2 + bx + c$ can not be factorised further	$\dfrac{A}{x-a} + \dfrac{Bx + c}{x^2 + bx + c}$

In the above table, A, B and C are real numbers to be determined suitably.

7. Definite Integral – The definite integral of $f(x)$ between the limits a to b i.e. in the interval [a,b] is denoted

by $\displaystyle\int_a^b f(x) \, dx$ and is defined as follows. $\displaystyle\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$ where $\displaystyle\int f(x) \, dx = F(x)$

8. General Properties of Definite Integrals –

Prop. I $\displaystyle\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

Prop. II $\displaystyle\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$

Prop. III $\displaystyle\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ where $a < c < b$

Prop. IV $\displaystyle\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx$

In particualr $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$

Prop. V $\int_0^{2a} f(x)\ dx$

Prop. V $\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx,$ if $f(x)$ is even function

$\int_{-a}^a f(x)\ dx = 0,$ if $f(x)$ is odd function

Prop. VI $\int_0^{2a} f(x)\ dx = 2\int_0^a f(x)\ dx + \int_0^a f(2a-x)\ dx$

Prop. VII $\int_0^{2a} f(x)\ dx = 2\int_0^a f(x)\ dx,$ if $f(2a-x)=f(x)$

$\int_0^{2a} f(x)\ dx = 0,$ if $f(2a-x)=-f(x)$

9. Definite Integral as the limit of a sum

$\int_a^b f(x)\ dx = \underset{h\to 0}{Lim}\ h\ [f(a)+f(a+h)+f(a+2h)+\cdots+f<a+(n-1)h)]$

or $\int_a^b f(x)\ dx = \underset{h\to 0}{Lim}\ h\ [f(a+h)+f(a+2h)+f(a+3h)+\cdots+f(a+nh)]$

where, $h = \dfrac{b-a}{n}$

$\dfrac{d}{dx}\int_{u(x)}^{v(x)} f(t)\ dt = f\ \{v(x)\}\dfrac{d}{dx}\ v(x) - f\ \{u(x)\}\dfrac{d}{dx}\ u(x)$ this rule is called leibnitz's is Rule.

CONNECTING CONCEPTS

1. Integration is an operation on function

2. $\int[k_1 f_1(x)+k_2 f_2(x)+\ldots\ldots\ldots+k_n f_n(x)]dx$

$= k_1\int f_1(x)\ dx + k_2\int f_2(x)\ dx + \ldots\ldots\ldots + k_n\int f_n(x)\ dx$

3. All functions are not integrable and the integral of a function is not unique.

4. If a polynomial function of a degree n is integrated we get a polynomial of degree n + 1

4. **Integration by using standard formulae –**

1. $\int k\,dx = kx + c,$ k is constant

2. $\int kf(x)\ dx = k\int f(x)\ dx + c$

3. $\int(f_1(x)\pm f_2(x)]\ dx = \int f_1(x)dx \pm \int f_2(x)\ dx + c$

4. $\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + c\ (n\neq -1)$

5. $\int \dfrac{1}{x}\ dx = \log_e|x| + c$

6. $\int a^x dx = \dfrac{a^x}{\log_e a} + c,\ a>0$

7. $\int e^x dx = e^x + c$

5. $\int \sin x\, dx = -\cos x + c$

6. $\int \cos x\, dx = \sin x + c$

7. $\int \sec^2 x\, dx = \tan x + c$

8. $\int \cosec^2 x\, dx = -\cot x + c$

9. $\int \sec x \tan x\, dx = \sec x + c$

10. $\int \cosec x \cot x\, dx = -\cosec x + c$

11. $\int \tan x\, dx = \log |\sec x| + c = -\log |\cos x| + c$

12. $\int \cot x\, dx = \log |\sin x| + c$

13. $\int \sec x\, dx = \log |\sec x + \tan x| + c$

14. $\int \cosec x\, dx = \log |\cosec x - \cot x| + c$

15. $\int \dfrac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1} x + c \text{ or } -\cos^{-1} x + c$

16. 16. $\int \dfrac{1}{1 + x^2}\, dx = \tan^{-1} x + c \text{ or } -\cot^{-1} x + c$

17. $\int \dfrac{1}{x\sqrt{x^2 - 1}}\, dx = \sec^{-1} x + c \quad \text{or} - \cosec^{-1} x + c$

18. 18. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + c$

19. 19. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a} \log \left|\dfrac{x - a}{x + a}\right| + c,\ x > a$

20. 20. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \log \left|\dfrac{a + x}{a - x}\right| + c,\ x < a$

21. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + c$

22. $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log \left|x + \sqrt{a^2 + x^2}\right| + c$

23. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \log \left|x + \sqrt{x^2 - a^2}\right| + c$

24. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} \sec^{-1}\left(\dfrac{x}{a}\right) + c$

25. $\int \sqrt{a^2 - x^2}\, dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{1}{2} a^2 \sin^{-1}\left(\dfrac{x}{a}\right) + c$

26. $\int \sqrt{x^2 + a^2} \ dx = \dfrac{x}{2} \sqrt{x^2 + a^2} + \dfrac{1}{2} a^2 \log \left| x + \sqrt{x^2 + a^2} \right| + c$

27. $\int \sqrt{x^2 - a^2} \ dx = \dfrac{x}{2} \sqrt{x^2 - a^2} - \dfrac{1}{2} a^2 \log \left| x + \sqrt{x^2 - a^2} \right| + c$

28. $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + c$

29. Use of Trigonometric Identities in Integration.

(i) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

(ii) $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$, $\cos^3 x = \dfrac{3\cos x + \cos 3x}{4}$

(iii) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
$2 \sin A \sin B = \cos (A - B) + \cos (A + B)$

(iv) $\sin x = 2 \sin \left(\dfrac{x}{2} \right) \cdot \cos \left(\dfrac{x}{2} \right)$

30.(i) $1 + 2 + 3 + \cdots\cdots + n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = \dfrac{n(n+1)(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^2 + \cdots\cdots + n^3 = \left[\dfrac{n(n+1)}{2} \right]^2$

(iv) $a + (a + d) + (a + 2d) + \cdots\cdots + [a + (n-1)d] = \dfrac{n}{2} [2a + (n-1)d]$

(v) $a + ar + ar^2 + \cdots\cdots + ar^{n+1} = \dfrac{a(r^n - 1)}{r - 1}$

7 INTEGRALS

1. **Integration** – The process of finding the function $f(x)$ whose differential coeffiicient w.r.t. 'x', denoted by $F(x)$ is given, is called the integration of $f(x)$ w.r.t. x and is written as $\int F(x)\, dx = f(x)$

    Thus, integration is an inverse process of differentiation or integration is anti of differentiation.

    The differential coefficient of a constant is zero. Thus if c is an arbitrary constant independent of x. then

    $\dfrac{d}{dx}[f(x)+c] = F(x)$ Thus $\int F(x)\, dx = f(x)+c$

    The arbitrary constant c is called the constant of integration.

2. **Integration by Substitution**

    (a) **To evaluate the integral** $\int f(ax+b)\, dx$

    Put $ax + b = t$, so that $adx = dt$ *i.e.,* $dx = \dfrac{1}{a} dt$

    $\int f(ax+b)\, dx = \int f(t) \cdot \dfrac{1}{a} dt = \dfrac{1}{a} F(t)$, where $\int f(t)\, dt = F(t) = F(ax+b)$

    If a function is not in some suitable form to find the integration, then we transform it into some suitable form by changing the independent variable x to t by substituting $x = g(t)$.

    Consider $\qquad I = \int f(x)\, dx$

    Put $\qquad\qquad x = g(t)$, so that $\dfrac{dx}{dt} = g'(t)$

    We write $\qquad dx = g'(t)\, dt$

    Thus $\qquad\qquad I = \int f(x) \cdot dx = \int f(g(t))\, g'(t)\, dt$

    But it is very important to guess, what will be the useful substitution.

    (b) $\int \dfrac{f'(x)}{f(x)} dx = \log f(x) + c$

    (c) $\int [f(x)]^n f'(x)\, dx = f(x)^{n+1} / (n+1) + c$

    (d) Some important substitutions

function	Substitutions
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

3. **Trigonometrical transformations** – For the integration of the trigonometrical products such as $\sin^2 x$, $\cos^2 x$, $\sin^3 x$, $\cos^3 x$, $\sin ax \cos bx$ etc. they are expressed as the sum or difference of the sines and cosines of multiples of angles.

4. Integration of Some Special Integrals –

(a) For $\int \dfrac{dx}{ax^2 + bx + c}$, $\int \dfrac{dx}{\sqrt{ax^2 + bx + c}}$ and $\int \sqrt{ax^2 + bx + c}\ dx$

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a\left[\left(x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right] = a\left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]$$

Put $x + \dfrac{b}{2a} = t$, $\therefore\ dx = dt$, $\dfrac{4ac - b^2}{4a^2} = \pm k^2$, $ax^2 + bx + c$ changes to $t^2 + k^2$, $t^2 - k^2$ or $k^2 - t^2$

(b) For $\int \dfrac{(px + q)\ dx}{ax^2 + bx + c}$, $\int \dfrac{(px + q)\ dx}{\sqrt{ax^2 + bx + c}}$, $\int (px + q)\ \sqrt{(ax^2 + bx + c)}\ dx$

Put $px + q = A\ \dfrac{d}{dx}\ (ax^2 + bx + c) + B$

Compare the two sides and find the value of A and B.

Thus $\int \dfrac{px + q}{ax^2 + bx + c} dx = \int \dfrac{A \dfrac{d}{dx}(ax^2 + bx + c) + B}{(ax^2 + bx + c)}$

$$= A\int \dfrac{\dfrac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)} dx + B\ \int \dfrac{dx}{(ax^2 + bx + c)}$$

Similarly $\int \dfrac{px + q}{\sqrt{ax^2 + bx + c}} dx = A\ \int \dfrac{\dfrac{d}{dx}(ax^2 + bx + c)}{\sqrt{ax^2 + bx + c}} dx + B\int \dfrac{dx}{\sqrt{ax^2 + bx + c}}$

same as do $\int (px + q)\ \sqrt{ax^2 + bx + c}\ dx$.

(c) For $\int \dfrac{dx}{(x + k)\ \sqrt{ax^2 + bx + c}}$ put $x + k = \dfrac{1}{t}$

(d) For $\int \dfrac{dx}{\sqrt{(x - \alpha)(x - \beta)}}$, $\int \sqrt{\dfrac{x - \alpha}{\beta - x}}\ dx$

$\int \sqrt{(x - \alpha)(x - \beta)}\ dx$, Put $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

(e) For $\int \dfrac{dx}{a + b \cos x}$, $\int \dfrac{dx}{a + b \sin x}$, $\int \dfrac{dx}{a + b \cos x + q \sin x}$

$\sin x = \left(2 \tan \dfrac{x}{2} \right) \Big/ \left(1 + \tan^2 \dfrac{x}{2} \right)$, $\cos x = \left(1 - \tan \dfrac{x}{2} \right) \Big/ \left(1 + \tan^2 \dfrac{x}{2} \right)$ then put $\tan x/2 = t$

(f) For $\int \dfrac{p \cos x + q \sin x}{a + b \cos x + b \sin x}\ dx$

Put $p \cos x + q \sin x = A\ (a + b \cos x + b \sin x) + B$ differential of $(a + b \cos x + b \sin x) + C$

A, B and C can be calculated by equating the coefficients of cos x. sin x and the constant terms.

5. Integration by parts $\int u \cdot v\ dx = u \cdot \int v\ dx - \int \left[\dfrac{du}{dx} \cdot \int v\,dx \right] dx$

i.e., the integral of the product of two functions = (first function) × (Integral of the second function – Integral of {(dfferential of first function) x (Integral of second function)}

This formula is called integration by parts.

6. Partial Integration – To Evaluate $\displaystyle\int \dfrac{P(x)}{Q(x)}\, dx$

The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors.

If $\dfrac{P(x)}{Q(x)}$ is improper fraction, i.e., degree of numerator is equal or greater than the degree of denominator. Then first we reduce in proper rational function as $\dfrac{P(x)}{Q(x)} = T(x) + \dfrac{P_1(x)}{Q(x)}$ where T (x) is a polynomial in x

and $\dfrac{P_1(x)}{Q(x)}$ is a proper rational function.

After this, the integration can be carried out easily using the already known methods. The following Table 7.1 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.1

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px - q}{(x-a)\,(x-b)},\ a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$
2.	$\dfrac{px + q}{(x-a)^2}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-b)^2}$
3.	$\dfrac{px^2 + qx + r}{(x-a)\,(x-b)\,(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
4.	$\dfrac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$
5.	$\dfrac{px^2 + qx + r}{(x-a)\,(x^2+bx+c)}$ Where $x^2 + bx + c$ can not be factorised further	$\dfrac{A}{x-a} + \dfrac{Bx+c}{x^2+bx+c}$

In the above table, A, B and C are real numbers to be determined suitably.

7. Definite Integral – The definite integral of f(x) between the limits a to b i.e. in the interval [a,b] is denoted

by $\displaystyle\int_a^b f(x)\, dx$ and is defined as follows. $\displaystyle\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$ where $\displaystyle\int f(x)\, dx = F(x)$

8. General Properties of Definite Integrals –

Prop. I $\displaystyle\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Prop. II $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Prop. III $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where $a < c < b$

Prop. IV $\displaystyle\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

In particualr $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Prop. V $\int_0^{2a} f(x)\,dx$

Prop. V $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even function

$\int_{-a}^a f(x)\,dx = 0$, if $f(x)$ is odd function

Prop. VI $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Prop. VII $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x)=f(x)$

$\int_0^{2a} f(x)\,dx = 0$, if $f(2a-x)=-f(x)$

9. Definite Integral as the limit of a sum

$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a)+f(a+h)+f(a+2h)+\cdots+f<a+(n-1)h)]$

or $\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a+h)+f(a+2h)+f(a+3h)+\cdots+f(a+nh)]$

where, $h = \dfrac{b-a}{n}$

$\dfrac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\dfrac{d}{dx}v(x) - f\{u(x)\}\dfrac{d}{dx}u(x)$ this rule is called leibnitz's is Rule.

CONNECTING CONCEPTS

1. Integration is an operation on function

2. $\int [k_1 f_1(x) + k_2 f_2(x) + \cdots\cdots + k_n f_n(x)]\,dx$

$= k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \cdots\cdots + k_n \int f_n(x)\,dx$

3. All functions are not integrable and the integral of a function is not unique.
4. If a polynomial function of a degree n is integrated we get a polynomial of degree n + 1
4. **Integration by using standard formulae –**

1. $\int k\,dx = kx + c$, k is constant

2. $\int k f(x)\,dx = k \int f(x)\,dx + c$

3. $\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx + c$

4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c\,(n \neq -1)$

5. $\int \dfrac{1}{x}\,dx = \log_e |x| + c$

6. $\int a^x\,dx = \dfrac{a^x}{\log_e a} + c,\ a > 0$

7. $\int e^x\,dx = e^x + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \mathrm{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \mathrm{cosec}\, x \cot x \, dx = -\mathrm{cosec}\, x + c$

11. $\int \tan x \, dx = \log|\sec x| + c = -\log|\cos x| + c$

12. $\int \cot x \, dx = \log|\sin x| + c$

13. $\int \sec x \, dx = \log|\sec x + \tan x| + c$

14. $\int \mathrm{cosec}\, x \, dx = \log|\mathrm{cosec}\, x - \cot x| + c$

15. $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c \ \text{ or } -\cos^{-1} x + c$

16. 16. $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c \ \text{ or } -\cot^{-1} x + c$

17. $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c \ \text{ or } -\mathrm{cosec}^{-1} x + c$

18. 18. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + c$

19. 19. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a} \log \left|\dfrac{x-a}{x+a}\right| + c, \ x > a$

20. 20. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log \left|\dfrac{a+x}{a-x}\right| + c, \ x < a$

21. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + c$

22. $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{a^2+x^2}\right| + c$

23. $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + c$

24. $\int \dfrac{dx}{x\sqrt{x^2-a^2}} = \dfrac{1}{a} \sec^{-1}\left(\dfrac{x}{a}\right) + c$

25. $\int \sqrt{a^2-x^2}\, dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{1}{2} a^2 \sin^{-1}\left(\dfrac{x}{a}\right) + c$

26. $\int \sqrt{x^2 + a^2}\ dx = \dfrac{x}{2} \sqrt{x^2 + a^2} + \dfrac{1}{2}\ a^2 \log \left| x + \sqrt{x^2 + a^2}\right| + c$

27. $\int \sqrt{x^2 - a^2}\ dx = \dfrac{x}{2} \sqrt{x^2 - a^2} - \dfrac{1}{2}\ a^2 \log \left| x + \sqrt{x^2 - a^2}\right| + c$

28. $\int e^x\ [f(x) + f'(x)]\ dx = e^x\ f(x) + c$

29. Use of Trigonometric Identities in Integration.

(i) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

(ii) $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$, $\cos^3 x = \dfrac{3\cos x + \cos 3x}{4}$

(iii) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
$2 \sin A \sin B = \cos (A - B) + \cos (A + B)$

(iv) $\sin x = 2 \sin \left(\dfrac{x}{2}\right) \cdot \cos \left(\dfrac{x}{2}\right)$

30.(i) $1 + 2 + 3 + \cdots\cdots + n = \dfrac{n\ (n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = \dfrac{n\ (n+1)\ (2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^2 + \cdots\cdots + n^3 = \left[\dfrac{n\ (n+1)}{2}\right]^2$

(iv) $a + (a + d) + (a + 2d) + \cdots\cdots + [a + (n - 1)\ d] = \dfrac{n}{2}\ [2a + (n - 1)\ d]$

(v) $a + ar + ar^2 + \cdots\cdots + ar^{n+1} = \dfrac{a(r^n - 1)}{r - 1}$

# 8 APPLICATION OF THE INTEGRALS

Area Under Simple Curves

1. Let us find the area bounded by the curve y = f (x), x-axis and the ordinated x = a and x = b. Consider the area under the curve as composed of large number of thin vertical strips let there be an arbitary strip of hieght y and width dx. Area of elementary strip dA= ydx, where y = f (x). Total Area A of the region between x-axis. ordinated x = a, x = b and the curve y = f (x) = Sum of areas of elementry thin strips across the region PQML
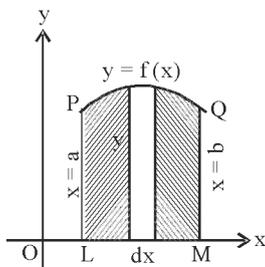
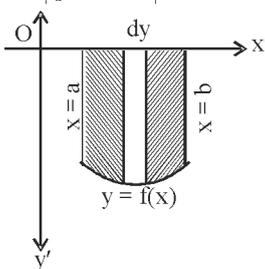$$A = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

2. The area A of the region bounded by the curve x = g (y), y-axis and the lines y = c and y = d is given by

$$A = \int_c^d x\,dy$$
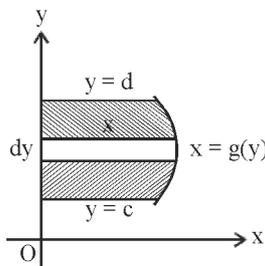
3. If the curve under consideration lies below x-axis, then f (x) < 0 from x = a to x = b, the area bounded by the curve y = f (x), and the ordinates x = a, x = b and x-axis is negative. But the numerical value of the area is to be taken into consideration. Then Area= $\left| \int_a^b f(x)\,dx \right|$

4. Let some portion of the curve is above x-axis and some portion is below x-axis. Let A_1 be the area below x-axis and A_2 be the area above of x-axis. Therefore Area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$.

$$A = |A_1| + A_2$$
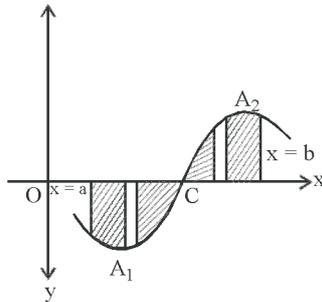
Area between Two curves

5. Let the two curves be $y = f(x)$ and $y = g(x)$. Suppose these curves intersect at $x = a$ and $x = b$. Consider the elementary strip of height y where $y = f(x) - g(x)$ with width dx
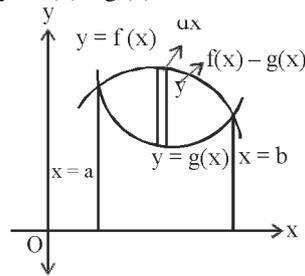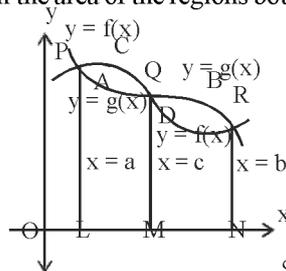

$\therefore \quad da = y dx$

$$\Rightarrow \quad A = \int_a^b \big(f(x) - g(x)\big)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

i.e. A= Area bounded by the curve $y = f(x)$ – Area bounded by the curve $y = g(x)$

6. If the two curves $y = f(x)$ and $y = g(x)$ intersects at $x = a$, $x = c$ and $x = b$ such that $a < c < b$. If $f(x) > g(x)$ in [a, c] and $f(x) < g(x)$ in [c, b], Then the area of the regions bounded by curve.

$$= \text{Area of the region PAQCP} + \text{Area of the region QDRBQ} = \int_a^c \big(f(x) - g(x)\big)dx + \int_c^b \big(g(x) - f(x)\big)dx$$

9 DIFFERENTIAL EQUATIONS

1. **Differential Equation** – An equation containing an independent variable dependent variable and differential coefficient of dependent variable with respect to independent variable is called a differential equation.

   e.g. $\dfrac{dy}{dx} + 2xy = x^3$ and $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2$

2. **Order of a differential Equation** – The order of a differential equation is the order of the highest order derivative appearing in the equation.

3. **Degree of a differential Equation** – The degree of a differential equation is the degree of the highest order derivative when differential coefficients are made free from radicals and fractions.

4. **Solution of a differential Equation** – The solution of a differential equation is a relation between the variables involved, not involving the differential coefficients, such that this relation and derivatives obtained form it satisfy the given differential equation.

5. **General Solution** – The solution which contains as many as orbirary constants as the order of the differential equation is called the general solution of the differential equation.

6. **Particular Solution** – Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

7. **Equations in variable separable form** – If the differential equation can be reduced to the form $f(x)\,dx = g(y)\,dy$ we say that the variables have been separated on integrating both sides of this reduced form, we get the general solution of the differential equation.

   $\int f(x)\,dx = \int g(y)\,dy + c$

8. **Equations Reducible to variable separable form** – Differential equations of the form $\dfrac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = v$

9. **Homogeneous Differential Equation** – A function $f(x,y)$ is called a homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non zero constant λ.

   A differential equation of the form $\dfrac{dy}{dx} = F(x, y)$ is said to be homogeneous if $F(x, y)$ is a homogeneous function of degree zero. To solve such a homogenous differential equation of the type

   $\dfrac{dy}{dx} = F(x) = g\left(\dfrac{y}{x}\right)$ ...(i)

   (i) Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in equation (i), we get reduces to the form $v + x\dfrac{dv}{dx} = g(v)$

   $\Rightarrow \quad x \times \dfrac{dv}{dx} = g(v) - v$

   Now, on separating the variables, we get

   $\dfrac{dv}{g(x) - v} = \dfrac{dx}{x}$

   Integrate both sides to obtain the solution in terms of v and x.

   Replace v by $\dfrac{y}{x}$ in the solution obtained to obtain the solution in terms of x and y.

If the homogeneous differential equation is in the form $\dfrac{dy}{dx} = F(x, y)$, where $F(x, y)$ is homogeneous function of degree, then we make substitution $\dfrac{x}{y} = v$ i.e., $x = vy$ and the proceed further to find the general solution as discussed above by writting $\dfrac{dx}{dy} = F(x, y) = h\left(\dfrac{x}{y}\right)$

10. **Linear differential Equations** – A differential equation is known as first order linear differential equation, if the dependent variable y and its derivative are related as $\dfrac{dy}{dx} + Py = Q$, where P and Q are constant or functions of x.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\dfrac{dy}{dx} + Py = Q$ and obtain P and Q.

(ii) Find integrating factor, I.F. $= e^{\int pdx}$

(iii) Multiply both sides of equation in (i) by I.F.

(iv) Integrate both sides of the equation obtained in (iii) w.r.t. x to obtain
$$y(I.F.) = \int Q.(I.F.) dx + C$$
This gives the required solution.

In case, the first order linear differential equation is in the form $\dfrac{dx}{dy} + P_1 x = Q_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F. $= e^{\int P_1 dy}$ and the solution of the differential equation is given by $x . (I. F) = \int (Q_1 \cdot I.F.) dy + C$

CONNECTING CONCEPTS

1. **Formation of Differential Equations** – Formation of a differential from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation representing a family of curves, contains n arbitrary constants, then we differentiable the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

2. **Methods of solving a differential equation of the type** $\dfrac{dy}{dx} = f(x)$ – To solve this type of differential equations, first we write the differential equation as $dy = f(x) dx$
Then integrate boht sides with respect t x to obtain the solution
$$\int dy = \int f(x) dx + C$$
or $\qquad y = \int f(x) dx + C$

3. **Differential Equations of the type** $\dfrac{dy}{dx} = f(y)$ – To solve this type of differential equations, first we write in the form of $dx = \dfrac{1}{f(y)} dy$ them integrate both sides to obtain the general solution
$$\Rightarrow \int dx = \int \dfrac{1}{f(y)} dy + c \text{ or } x = \int \dfrac{1}{f(y)} dy + c$$

4. Differential Equations of the type $\dfrac{d^2 y}{dx^2} = f(x)$

(i) Integrate both sides of the differential equation in (i) with respect to x to obtain a first order first degree differential equation.

(ii) Integrate both sides of the first order differential equation obtained in (ii) with respect to x.

10 VECTOR ALGEBRA

1. **Vector** – A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration.
   AB is a directed line segment. It is a vector AB and its direction is from A to B.

    A————————————→B

    Initial Points – The point A where from the vector AB starts is known as initial point.
    Terminal Point – The point B, where it ends is said to be the terminal point.
    Magnitude – The distance between initial point and terminal point of a vector is the magnitude or length of the vector AB. It is denoted by | AB | or AB.

2. **Position Vector** – Consider a point p (x, y, z) in space. The vector OP with initial point, origin O and terminal point P, is called the position vector of P.
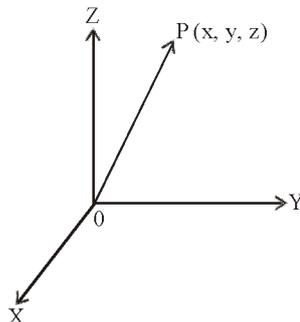
3. **Types of Vectors**
    (i) **Zero Vector Or Null Vector** – A vector whose initial and terminal points coincide is known as zero vector (O).
    (ii) **Unit Vector** – A vector whose magnitude is unity is said to be unit vector. It is denoted as $\hat{a}$ so that $|\hat{a}| = 1$.
    (iii) **Co-initial Vectors** – Two or more vectors having the same initial point are called co-initial vectors.
   (iv) **Collinear Vectors** – If two or more vectors are parallel to the same line, such vectors are known as collinear vectors.
    (v) **Equal Vectors** – If two vectors a and b have the same magnitude and direction regardless of the positions of their initial points, such vectors are said to be equal *i.e.,* a = b .
    (vi) **Negative of a vector** – A vector whose magnitude is same as that of a given vector AB , but the direction is opposite to that of it, is known as negative of vector AB *i.e.,* BA = – AB

4. **Sum of Vectors**
    (i) **Sum of vectors a and b** let the vectors a and b be so positioned that initial point of one coincides with terminal point of the other. If a = AB , b = BC . Then the vector a + b is represented by the third side of △ ABC. *i.e.,* AB + BC = AC   ...(i)
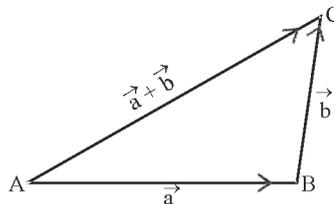
This is known as the triangle law of vector addition.

Further $AC = -CA$

$$AB + BC = -CA \qquad \therefore \quad AB + BC + CA = 0$$

when sides of a triangle ABC are taken in order i.e. initial and terminal points coincides. Then
$$AB + BC + CA = 0$$

(ii) **Parallelogram law of vector addition** – If the two vectors a and b are represented by the two adjacent sides OA and OB of a parallelogram OACB, then their sum a + b is represented in magnitude and direction by the diagonal OC of parallelogram through their common point O *i.e.,* $OA + OB = OC$
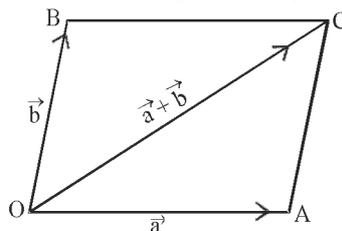

5. **Multiplication of Vector by a Scalar** – Let a be the given vector and λ be a scalar, then product of λ and
    $a = \lambda a$

    (i) when λ is +ve, then a and λa are in the same direction.

    (ii) when λ is –ve. then a and λa are in the opposite direction. Also $\left| \lambda \, \vec{a} \right| = \left| \lambda \right| \left| \vec{a} \right|$.

6. **Components of Vector** – Let us take the points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) on the coordinate axes OX, OY and OZ respectively. Now, | OA | = 1, | OB | = 1 and | OC | = 1, Vectors OA, OB and OC each having magnitude 1 is known as unit vector. These are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$.
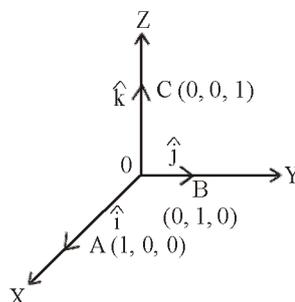
Consider the vector OP , where P is the point (x, y, z). Now OQ, OR, OS are the projections of OP on coordinates axes.

$$\therefore \quad OQ = x, OR = y, OS = z \qquad \therefore \quad OQ = x\hat{i}, \quad OR = y\hat{j} \quad , \quad OS = z\hat{k}$$
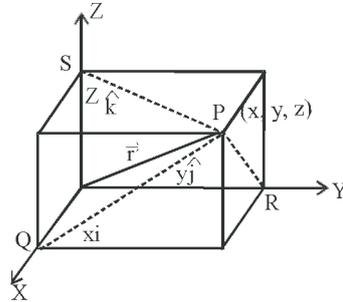
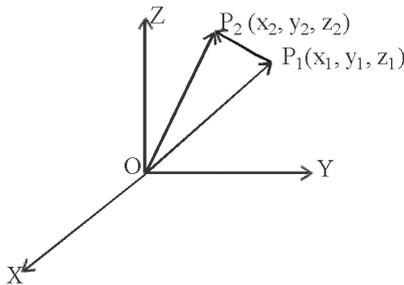$$\Rightarrow \qquad \overrightarrow{OP} = x\hat{i}, + y\hat{j}, + z\hat{k} \;, \qquad |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$$

x, y, z are called the scalar components and $x\hat{i}, y\hat{j}$, $z\hat{k}$ are called the vector components of vector OP .

7. **Vector joining two points** – Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2 z_2)$ be the two points. Then vector joining the points P_1 and P_2 is P_1P_2 . Join P_1, P_2 with O. Now $OP_2 = OP_1 + P_1P_2$ (by triangle law)

$$\therefore \qquad P_1P_2 = OP_2 - OP_1$$

$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

8. **Section Formula**

    (i) A line segment PQ is divided by a point R in the ratio m : n internally *i.e.,* $\dfrac{PR}{RQ} = \dfrac{m}{n}$
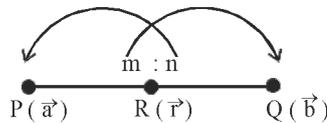
If a and b are the position vectors of P and Q then the position vector r of R is given by

$$\vec{r} = \frac{mb + na}{m + n}$$

If R be the mid-point of PQ, then $\vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$

    (ii) when R divides PQ externally, i.e., $|a \cdot b|\,\hat{n}$

Then $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$
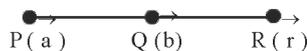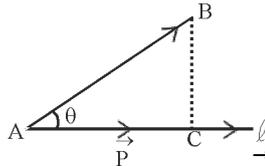
9. **Projection of vector along a directed line** – Let the vector AB makes an angle θ with directed line $\cdot$.
Projection of AB on $\cdot = |\overrightarrow{AB}|\cos\theta = \overrightarrow{AC} = \vec{p}$.

The vector $\vec{p}$ is called the projection vector. Its magnitudes is $\left|\vec{b}\right|$, which is known as projection of vector AB. The angle θ between AB and AC is given by

$\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|AB|\,|AC|}$, Now projection $AC = |AB|\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AC}|}$

$= \overrightarrow{AB} \cdot \left(\dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|}\right)$, If $\overrightarrow{AB} = \vec{a}$, then $\overrightarrow{AC} = \vec{a} \cdot \left(\dfrac{\vec{p}}{|\vec{p}|}\right) = \vec{a} \cdot \hat{p}$

Thus, the projection of $\vec{a}$ on $\vec{b} = \vec{a} \cdot \left(\dfrac{\vec{b}}{|\vec{b}|}\right) = \vec{a}\cdot\hat{b}$

10. **Scalar Product of Two Vectors (Dot Product)** – Scalar Product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

Where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$

(i) when $\theta = 0$, then $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}| = ab$ Also $\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}| = a.a = a^2$

$\therefore$ $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(ii) when $\theta = \dfrac{\pi}{2}$, then $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\dfrac{\pi}{2} = 0$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

11. **Vector Product of two Vectors (Cross Product)** – The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$ is defined as

$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\cdot\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.

Unit vector $\hat{n}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}\cdot\vec{b}$ and $\hat{n}$ form a right handed orthogonal system.

(i) If $\theta = 0$, then $\vec{a}\times\vec{b} = 0$, $\therefore \vec{a}\times\vec{a} = 0$

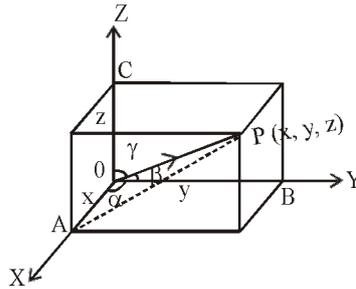and $\therefore \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(ii)   If $\theta = \Pi/_2$, then $\vec{a}\times\vec{b} = |\vec{a}\cdot\vec{b}|\hat{n}$

$\hat{i}\times\hat{j} = \hat{k}, \ \hat{j}\times\hat{k} = \hat{i}, \ \hat{k}\times\hat{i} = \hat{j}$

Also, $\hat{j}\times\hat{i} = -\hat{k}, \ \hat{k}\times\hat{j} = -\hat{i}$ and $\hat{i}\times\hat{k} = \hat{j}$

1. **Direction Cosines** – Let OX, OY, OZ be the positive coordinate axes, P (x, y, z) by any point in the space. Let OP makes angles α, β, γ with coordinate, axes OX, OY, OZ. The angle α, β, γ are known as direction angles, cosine of these angles *i.e.,*

cos α, cos β, cos γ are called direction cosines of line OP. these direction cosines are denoted by ·, m, n

i.e., · = cos α, m = cos β, n = cos γ

2. **Relation Between, l, m, n and Direction Ratios** –

The perpendiculars PA, PB, PC are drawn on coordinate axes OX, OY, OZ reprectively. Let | OP | = r

In $\triangle$ OAP, $\angle$ A = 90°, cos α = $\dfrac{x}{r}$ = ·, $\therefore$ x = ·r, In $\triangle$ OBP. $\angle$ B = 90°, cos β = $\dfrac{y}{r}$ = m $\therefore$ y = mr

In $\triangle$ OCP, $\angle$ C = 90°, cos γ = $\dfrac{z}{r}$ = n, $\therefore$ z = nr

Thus the coordinates of P may b expressed as (·r, mr, nr)

Also, OP^2 = x^2 + y^2 + z^2, r^2 = (lr)^2 + (mr)^2 + (nr)^2 $\Rightarrow$ ·^{2} + m^2 + n^2 = 1

Set of any there numbers, which are proportional to direction cosines are called direction ratio of the vactor. Direction ratio are denoted by a, b and c.

The numbers ·r mr and nr, proportional to the direction cosines, hence, they are also direction ratios of vector OP .

3. **Properties of Vector Addition** –

1. For two vectors $\vec{a}, b$ the sum is commutative *i.e.,* $\vec{a} + b = b + \vec{a}$

2. For three vectors $\vec{a}, b$ and c , the sum of vectors is associative i.e.,

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

4. **Additive Inverse of Vector** a – If there exists vector – a such that $\vec{a} + (-\vec{a}) = \vec{a} - \vec{a} = 0$ then – a is called the additure inverse of a.

5. **Some Properties** – Let a = a$_1$ $\hat{i}$ + a$_2$ $\hat{j}$ + a$_3$ k and b = b$_1$ $\hat{i}$ + b$_2$ $\hat{j}$ + b$_3$ $\hat{k}$

(i) $\vec{a} + \vec{b}$ = (a$_1$ $\hat{i}$ + a$_2$ $\hat{j}$ + a$_3$ $\hat{k}$) + (b$_1$ $\hat{i}$ + b$_2$ $\hat{j}$ + b$_3$ $\hat{k}$) = (a$_1$ + b$_1$) $\hat{i}$ + (a$_2$ + b$_2$) $\hat{j}$ + (a$_3$ + b$_3$) $\hat{k}$

(ii) $\vec{a} = \vec{b}$ or (a$_1$ $\hat{i}$ + a$_2$ $\hat{j}$ + a$_3$ $\hat{k}$) = (b$_1$ + b$_2$ $\hat{j}$ + b$_3$ $\hat{k}$) $\Rightarrow$ a$_1$ = b$_1$, a$_2$ = b$_2$, a$_3$ = b$_3$

(iii)(iii) $\lambda \vec{a} = \lambda$ (a$_1$ $\hat{i}$ + a$_2$ $\hat{j}$ + a$_3$ $\hat{k}$) = (λa$_1$)$\hat{i}$ + (λa$_2$)$\hat{j}$ + (λa$_3$)$\hat{k}$

(iv) a and b are parallel, if and only if there exists a non zero scalar λ such that b = $\lambda \vec{a}$

i.e., $\quad b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

$\therefore \quad b_1 = \lambda a_1, \; , b_2 = \lambda a_2, \; b_3 = \lambda a_3 \qquad \therefore \dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = \lambda$

6. Properties of scalar product of two vectors (Dot Product)

(i) $\cos \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \, |\vec{b}|}$

If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \qquad \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

Then, $\quad (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$, $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\vec{a} \cdot \vec{b} =$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \; |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} \qquad \therefore \cos \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \, |\vec{b}|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$

(ii) $\vec{a} \cdot \vec{b}$ is commutative *i.e.,* $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) If α is a scalar, then $(\alpha \vec{a}) \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\alpha \vec{b})$

7. Properties of Vector Product of two Vectors (Cross Product) –

(i) (a) If $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a} \times \vec{b} = 0$

(b) If $\vec{a} \cdot \vec{b}$, then $\vec{a} \times \vec{b} = 0$

(ii) $\vec{a} \times \vec{b}$ is not commutative

i.e. $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(iii) If $\vec{a}$ and $\vec{b}$ represent adjacent sides of a parallelogram, then its area $|\vec{a} \times \vec{b}|$

(iv) If $\vec{a}, \vec{b}$ represent the adjacent sides of a triangle, then its area $= \dfrac{1}{2} |\vec{a} \times \vec{b}|$

(v) Distributive property $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(a) If α be a scalar, then $\alpha (\vec{a} \times \vec{b}) = (\alpha \vec{a}) \times \vec{b} = \vec{a} \times (\alpha \vec{b})$

(b) If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

Then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

8. If α, β, γ are the direction angles of the vector $\vec{a} = \left(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \right)$. Then direction cosines of $\vec{a}$ are given as

$\cos \alpha = \dfrac{a_1}{|\vec{a}|}, \; \cos \beta = \dfrac{a_2}{|\vec{a}|}, \; \cos \gamma = \dfrac{a_3}{|\vec{a}|}$

9. **Scalar Product of Two Vectors (Dot Product)** – Scalar Product of two vectors a and b is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos \theta$$

where θ is the angle between a and b $\left(0 \leq \theta < \dfrac{\pi}{2}\right)$

(i) When $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$. Also $a \cdot a$ $a \cdot a = a^2$

$\therefore \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(ii) When $\theta = \dfrac{\pi}{2}$, $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos \dfrac{\pi}{2} = 0$

11 THREEE DIMENSIONAL GEOMETRY

GENERAL KEY CONCEPTS

1. **Distance Formula :** Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. **Section Formula :**
   (i) If a point R divides the line segment joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$ internally, then

   $$R\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

   (ii) If a point R divides the line segment joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$ externally, then

   $$R\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

3. **Mid-point Formula :** If R be the mid point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

   $$R\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

4. Centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

   $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

CONNECTING CONCEPTS

1. To locate the position of a point in three dimensional space, we consider a rectangular coordinate system of three mutually perpendicular lines as the coordinate axes. These axes are called x, y and z-axes.
2. The three planes determined by the pair of axes are the coordinate planes called XY, YZ and ZX-planes. The three coordinate planes divide the space into eight parts known as octants.
   The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances of the point P from the YZ, ZX and XY-plane.
   The co-ordinate of a point in three dimensional space are also the distances from the origin of the feet of the perpendicular drawn from the point on the respective co-ordinate axes.

3. The sign of the coordinates of a point is determined by the octant in which the point lies.

Octant / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

4. (i) Any point on x-axis is of the form (x, 0, 0)

(ii) Any point on y-axis is of the form (0, y, 0)

(iii) Any point on z-axis is of the form (0, 0, y)

5. The distance of the point (x, y, z) from the origin is given by $\sqrt{x^2 + y^2 + z^2}$

13 PROBABILITY

KEY CONCEPT INVOLVED

1. **Conditional Probability** – Let E and F be two events of a random experiment, then, the probability of occurance of E under the condition that F has alredy occured and P (F) ≠ 0 is called the conditional probability. It is denoted by P (E/F)

   The conditional probability P (E/F) is given by $P(E/F) = \dfrac{P(E \cap F)}{P(F)}$, When P (F) ≠ 0

   Properties of conditional probability –
   (i) If F be an event of a sample space s of an experiment, then P (S/F) = P (F/F) = 1
   If A and B are any two events of a sample space S and F is an event of s such that P (F) ≠ 0, then
   (ii) P (A ∪ B/F) = P(A/F) + P (B/F) – P (A ∩ B/F)
   IF A and B are disjoint event then P(A ∪ B/F) = P(A/F) + P (B/F)
   (iii) $P(\bar{E}/F) = 1 - P(E/F)$ or P (E'/F) = 1 – P (E/F)

2. **Multiplication Theorem On Probability** – Let E and F be two events associated with a sample space S. P(E ∩ F) denotes the probability of the event that both E and F occur, which is given by P (E ∩ F) = P (E) P (F/E) = P (F) P (E/F), provided P (E) ≠ 0 and P(F) ≠ 0

3. **Independent Event –**
   (i) Events E and F are independent if P (E ∩ F) = P (E) × P (F)
   (ii) Two events E and F are said to be independent if P (E/F) = P (E) and P (F/E) = P (F)
   provided P (E) ≠ 0 and P (F) ≠ 0
   (iii) Three events E, F and G are said to be independent or mutually independent if
   P (E ∩ F ∩ G) = P (E) P (F) P (G).

4. **Random Variable** – A random variable is a real valued function whose domain is the sample space of random experiment.

5. **Baye's Theorem** – let $E_1, E_2, \cdots\cdots\cdots\cdots, E_n$ be the x events forming a partition of sample space S i.e. $E_1, E_2, \cdots\cdots\cdots\cdots, E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup \cdots\cdots\cdots \cup E_n = S$ and A is any event of non – zero porbability, then $P(E_i/A) = \dfrac{P(E_i)\ P(A/E_i)}{\displaystyle\sum_{j=1}^{n} P(E_j) P(A/E_j)}$ for any i = 1, 2, 3,........ , n

6. **Bernoulli Trial** – Trials of a random experiment are said to be Bernoulli's trials, if they satisfy the following conditions :
   (i) The trials should be independent.
   (ii) Each trial has exactly two outcomes ex- success or falilure.
   (iii) The probability of success remains the same in each trial.
   (iv) Number of trials is finite.

7. **Mean of Random Variable** – let X be a random variable whose possible values are $x_1, x_2, \cdots\cdots, x_n$ if $P_1, P_2, \cdots\cdots, P_n$ are the corresponding probabilities, then mean of X,

$$\mu = \sum_{i=1}^{n} x_i \, p_i = E(X)$$

The mean of a random variables X is also called the expected value of X denoted by E (x).

8. **Variance of a Random Variable** – let X be a random variable with possible values $x_1\, x_2, \cdots\cdots x_n$ occur with probabilities are $p_1, p_2, \cdots\cdots p_n$ respectively.

   let $\mu = E(X)$ be the mean of X. The variance of X denoted by var (X) or σ^2_x is defined as

$$\text{Var (X) or } \sigma^2_x = \sum_{i=1}^{n} (x_i - \mu)^2 \, p_i \qquad = E(x_i - \mu)^2 \qquad = E(X^2) - [E(X)]^2$$

   Standard Deviation, $\sigma_x = \sqrt{\text{Var}(X)}$

9. **Probability function** – The probability of x success is denoted by p (X = x) or P(x) and is given by P (x) = $^{n}C_x \, q^{n-x} p^x$, $x = 0, 1, 2, \cdots\cdots, n$ and $q = 1 - P$

   The function P (x) is known as probability function of binomial distribution.

CONNECTING CONCEPTS

1. **Partition of a sample space** – A set of events $E_1, E_2, \cdots\cdots E_n$ is said to represent a partition of sample S if

   (i) $E_i \cap F_j = \phi \text{ if } i \ne j$, $i, j = 1, 2, \cdots\cdots, n$

   (ii) $E_1 \cup E_2 \cup E_3 \cup \cdots\cdots, \cup E_n = S$

   (iii) $P(E_i) > 0 \ \forall \ i = 1,2, \cdots\cdots, n$.

2. **Theorem of total Probability** – let $\alpha E_1, E_2, \cdots\cdots, E_n \gamma$ be a partition of sample spaces and each event has a non – zero probability If A be any event associated with S, then

$$P(A) = P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + P(E_3) P(A/E_3) + \cdots\cdots + P(E_n) P(A/E_n)$$

$$P(A) = \sum_{i=1}^{n} P(E_i) P(A/E_i)$$

3. **A Few Terminologies** –

   (i) **Hypothesis** – When Baye's theorem is applied the events $E_1, E_2, \cdots\cdots, E_n$ are said to be hypothesis x.

   (ii) **Priori Probability** – The Probabilities $P(E_1), P(E_2), \cdots\cdots, P(E_n)$ are called priori.

   (iii) **Posteriori Probability** – The conditional probability $P(E_i/A)$ is known as the posteriori probability of hypothesis E_i where $i = 1, 2, 3, \cdots\cdots, n$

4. **Probability Distribution of a Random Variable** – let real numbers $x_1, x_2, \cdots\cdots, x_n$ be the possible value of random variable and $p_1, p_2, \cdots\cdots, p_n$ be probability corresponding to each value of the random variable X. Then the probability distribution is

X:	x_1	x_2	$\cdots\cdots x_n$
P(X):	p_1	p_2	$\cdots\cdots p_n$.

   (i) $p_i > 0$ (ii) sum of probabilities $p_1 + p_2 + \cdots\cdots + p_n = 1$.

5. **Binomial Distribution** – Probability distribution of a number of successes, in an experiment consisting of n Bernoulli trials are obtained by Binomial expansion of $(q + p)_n$. Such a probability distribution is

X :	0	1	2	$\cdots\cdots$ r	$\cdots\cdots$ n
P(X):	$^{n}C_0 \, q^n$	$^{n}C_1 \, q^{n-1} P$	$^{n}C_2 \, q^{n-2} P^2$	$^{n}C_r \, q^{n-r} P^r$	$^{n}C_n \, P^n$

   This probability distribution is called binomial distribution with parameter n and p.

   Where, p is the probability of success in each trial and q is the probability of not sucess in each trial.

   $\therefore \qquad p + q = 1 , q = 1 - p$