CHAPTER

1 Relations and Functions

Relations

If A and B are two non-empty sets, then a relation R from A to B is a subset of $A \times B$.

Representation of a Relation

Roster form: In this form, we represent the relation by the set of all ordered pairs belongs to R.

Set-builder form: In this form, we represent the relation R from set A to set B as

$R = \{(a, b) : a \in A, b \in B$ and the rule which relate the elements of A and $B\}$.

Domain, Codomain and Range of a Relation

Let R be a relation from a non-empty set A to a non-empty set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the range of R. Also, the set B is called the codomain of relation R.

Thus, domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$

Types of Relations

Empty or Void Relation: As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.

Universal Relation: Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.

Identity Relation: The relation $I_A = \{(a, a): a \in A\}$ is called the identity relation on A.

Reflexive Relation: A relation R on a set A is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall a \in A \Rightarrow R$ is reflexive.

Symmetric Relation: A relation R on a set A is said to be symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

i.e. $a R b \Rightarrow bRa, \forall a, b \in$ A

Transitive Relation: A relation R on a set A is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad (a, c) \in R, \forall a, b, c \in A$

Equivalence Relation

A relation R on a set A is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

Functions

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

Domain, Codomain and Range of a Function

If f : $A \rightarrow B$ is a function from A to B, then

(*i*) the set A is called the domain of $f(x)$.

(*ii*) the set B is called the codomain of $f(x)$.

(*iii*) the subset of B containing only the images of elements of A is called the range of $f(x)$.

Number of Functions

Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

Number of One-One Functions

Let A and B are finite sets having m and n elements repectively, then the number of one-one functions from A to B is $\begin{cases} {}^nP_m, n \geq m \\ 0, n < m \end{cases}$

$$= \begin{cases} n(n-1)(n-2)...(n-(m-1)), & n \geq m \\ 0, & n < m \end{cases}$$

Number of Onto (or Surjective) Functions

Let A and B are finite sets having m and n elements respectively, then number of onto (or surjective) functions from A to B is

$$= \begin{cases} n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + ..., & n < m \\ n!, & n = m \\ 0, & n > m \end{cases}$$

Number of Bijective Functions

Let A and B are finite sets having m and n elements respectively, them number of bijective functions from A to B is

$$= \begin{cases} n!, \text{ if } n = m \\ 0, \text{ if } n > m \text{ or } n < m \end{cases}$$

Properties of Greatest Integer Function

(*i*) $[x + n] = n + [x], n \in I$

(*ii*) $[-x] = -[x], x \in I$

(*iii*) $[-x] = -[x] - 1, x \notin I$

(*iv*) $[x] \geq n \Rightarrow x \geq n, n \in I$

(*v*) $[x] > n \Rightarrow x \geq n + 1, n \in I$

(*vi*) $[x] \leq n \Rightarrow x < n + 1, n \in I$

(*vii*) $[x] < n \Rightarrow x < n, n \in I$

(*viii*) $[x + y] \geq [x] + [y]$

Important Points To Be Remembered

(*i*) Constant function is periodic with no fundamental period.

(*ii*) If $f(x)$ is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.

(*iii*) If $f(x)$ is periodic with period T, then $kf(ax + b)$ is periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.

Properties of Even and Odd Functions

(*i*) *gof* or *fog* is even, if both *f* and *g* are even or if *f* is odd and *g* is even or if *f* is even and *g* is odd.

(*ii*) *gof* or *fog* is odd, if both of *f* and *g* are odd.

(*iii*) If $f(x)$ is an even function, then $\frac{d}{dx}f(x)$ is an odd function and if $f(x)$ is an odd function, then $\frac{d}{dx}f(x)$ is an even function.

(*iv*) The graph of an even function is symmetrical about *Y*-axis.

(*v*) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.

(*vi*) An even function can never be one-one, however an odd function may or may not be one-one.

Properties of Inverse Function

(*a*) The inverse of a bijection is unique.

(*b*) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.

(*c*) The inverse of a bijection is also a bijection.

(*d*) If f & g are two bijections $f : A \to B$, $g : B \to C$ & *gof* exist, then the inverse of *gof* also exists and $(gof)^{-1} = f^{-1} og^{-1}$.

(*e*) The graph of f^{-1} obtained by reflecting the graph of f about the line $y = x$.

General

If x, y are independent variables, then :

(*a*) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell nx$

(*b*) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$ or $f(x) = 0$

(*c*) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(*d*) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

CHAPTER

2 Inverse Trigonometric Functions

Principal Values and Domains of Inverse Trigonometric/circular Functions

	Function	Domain	Range
(*i*)	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(*ii*)	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(*iii*)	$y = \tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(*iv*)	$y = \operatorname{cosec}^{-1} x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2};\ y \ne 0$
(*v*)	$y = \sec^{-1} x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi;\ y \ne \frac{\pi}{2}$
(*vi*)	$y = \cot^{-1} x$	$x \in R$	$0 < y < \pi$

Properties of Inverse circular Functions

P-1:

(*i*) $y = \sin(\sin^{-1} x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic.

(*ii*) $y = \cos(\cos^{-1} x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic.

(*iii*) $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic.

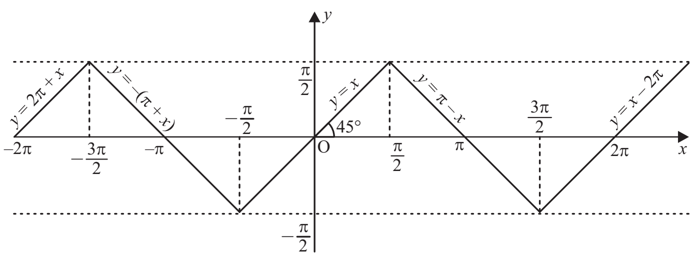
(*iv*) $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic.

(*v*) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, $|x| \ge 1$, $|y| \ge 1$, y is aperiodic.

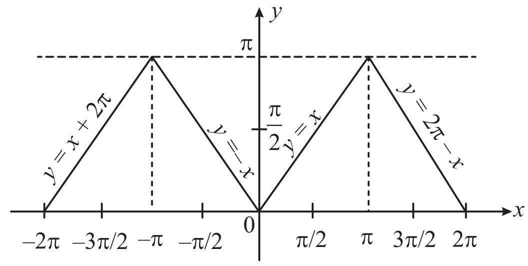
(*vi*) $y = \sec(\sec^{-1} x) = x$, $|x| \ge 1$; $|y| \ge 1$, y is aperiodic.

P-2:

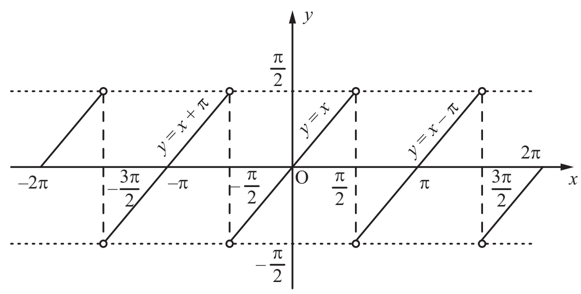
(*i*) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π.



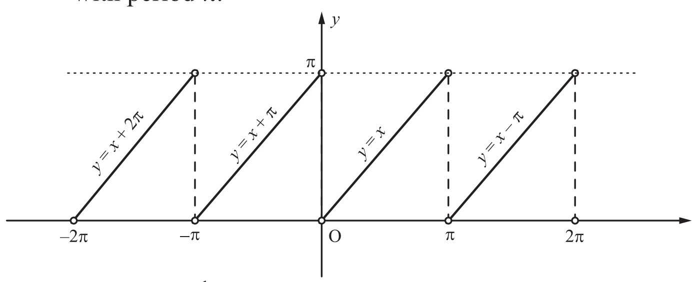
(*ii*) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, periodic with period 2π.



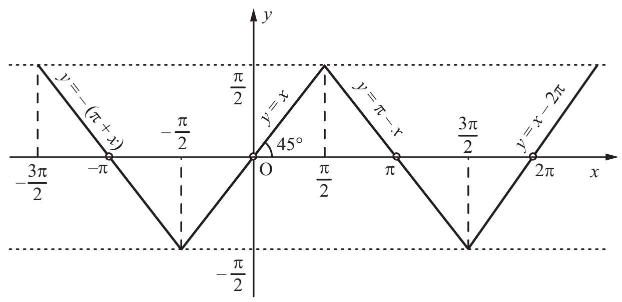
(*iii*) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$



(*iv*) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $n \in I$, $y \in (0, \pi)$, periodic with period π.

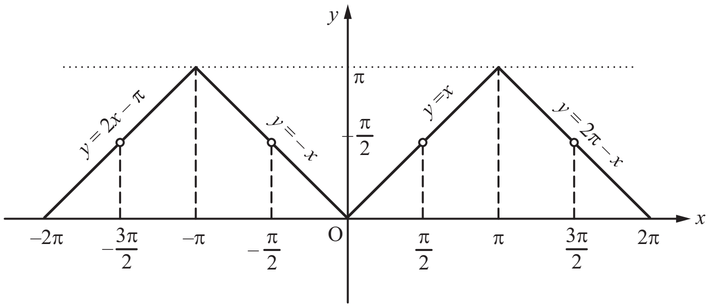


(*v*) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$, $x \in R - \{n\pi\}$, $n \in I$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is periodic with period 2π.



(*vi*) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

$$x \in R - \left\{(2n-1)\frac{\pi}{2}\right\}, n \in I,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



P-3:

(*i*) $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $\quad x \le -1, x \ge 1$

(*ii*) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $\quad x \le -1, x \ge 1$

(*iii*) $$\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x}; & x > 0 \\ \pi + \tan^{-1}\frac{1}{x}; & x < 0 \end{cases}$$

P-4:

(*i*) $\sin^{-1}(-x) = -\sin^{-1} x,\ -1 \le x \le 1$

(*ii*) $\tan^{-1}(-x) = -\tan^{-1} x,\ x \in R$

(*iii*) $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ -1 \le x \le 1$

(*iv*) $\cot^{-1}(-x) = \pi - \cot^{-1} x,\ x \in R$

(*v*) $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ x \le -1$ or $x \ge 1$

(*vi*) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,\ x \le -1$ or $x \ge 1$

P-5:

(*i*) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; $\quad -1 \le x \le 1$

(*ii*) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$; $\quad x \in R$

(*iii*) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$; $\quad |x| \ge 1$

P-6:

(*i*) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy}, & xy < 1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy}, & \text{where } x > 0, y > 0 \text{ and } xy > 1 \\ \frac{\pi}{2}, & \text{where } x > 0, y > 0 \text{ and } xy = 1 \end{cases}$$

(*ii*) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x-y}{1+xy}, & xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{where } x > 0, y > 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{where } x < 0, y < 0 \text{ and } xy < -1 \end{cases}$$

(*iii*) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$,

where $x \ge 0, y \ge 0$ & $(x^2 + y^2) < 1$

Note that: $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \frac{\pi}{2}$

(*iv*) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$,

where $x > 0, y > 0$ and $x^2 + y^2 > 1$.

Note that: $x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$.

(*v*) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

where $x > 0, y > 0$.

(*vi*) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$; $x, y \ge 0$

(*vii*) $\cos^{-1} x - \cos^{-1} y =$

$$\begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x > 0, y > 0 \text{ and } x < y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x > 0, y > 0 \text{ and } x > y \end{cases}$$

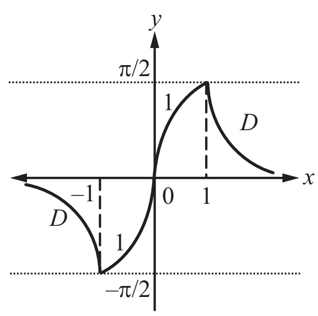
(*viii*) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x + y + z - xyz}{1 - xy - yz - zx}\right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$.

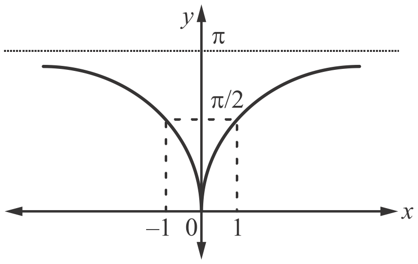
Note that: In the above results x & y are taken positive. In case if these are given as negative, we first apply *P*-4 and then use above results.

Simplified Inverse Trigonometric Functions

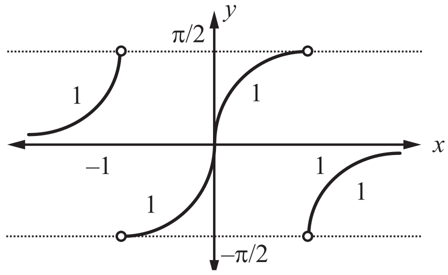
(*a*) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1} x & \text{if} & |x| \le 1 \\ \pi - 2\tan^{-1} x & \text{if} & x > 1 \\ -(\pi + 2\tan^{-1} x) & \text{if} & x < -1 \end{bmatrix}$$



(*b*) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x \text{ if } x \ge 0 \\ -2\tan^{-1} x \text{ if } x < 0 \end{cases}$

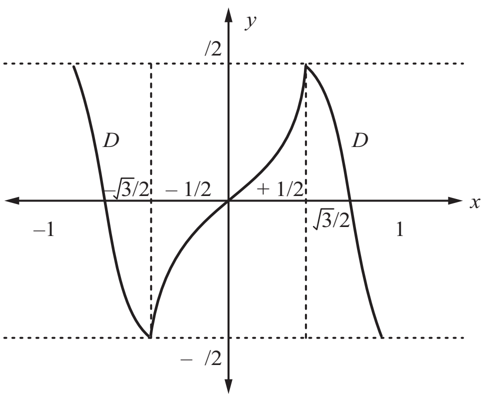


(*c*) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1} x) & \text{if } x > 1 \end{cases}$



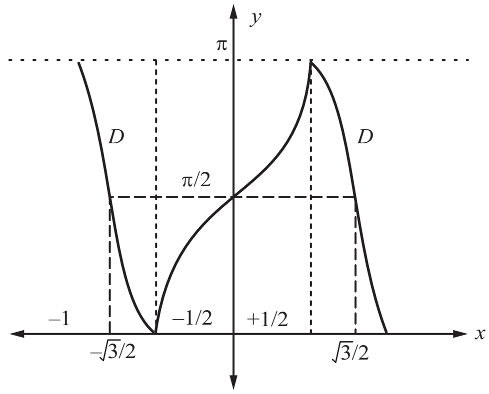
(*d*) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if } -1 \le x \le -\dfrac{1}{2} \\ 3\sin^{-1} x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}$$

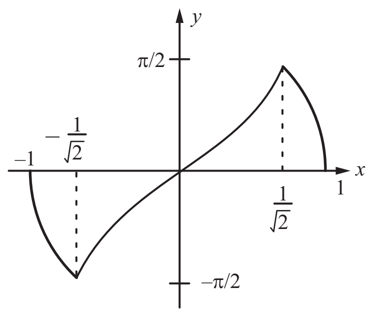


(*e*) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if } -1 \le x \le -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1} x & \text{if } \dfrac{1}{2} \le x \le 1 \end{cases}$$



(*f*) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -(\pi + 2\sin^{-1} x) & -1 \le x \le -\dfrac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \dfrac{1}{\sqrt{2}} \le x \le 1 \end{cases}$



CHAPTER

3 Matrices

Special Type of Matrices

1. **Row Matrix (Row vector):** $A = [a_{11}, a_{12}, \ldots a_{1n}]$ i.e., row matrix has exactly one row.
2. **Column Matrix (Column vector):** $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e., column matrix has exactly one column.
3. **Zero or Null Matrix:** ($A = O_{m\times n}$), an $m \times n$ matrix whose all entries are zero.
4. **Horizontal Matrix:** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.
5. **Vertical Matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$.
6. **Square Matrix:** (Order n) if number of rows = number of column, then matrix is a square matrix.

Key Note

- The pair of elements a_{ij} and a_{ji} are called Conjugate Elements.
- The elements $a_{11}, a_{22}, a_{33}, \ldots a_{mm}$ are called Diagonal Elements. the line along which the diagonal elements lie is called "Principal or Leading diagonal." The quantity Σa_{ii} = trace of the matrix written as, $t_r(A)$.

7. **Unit/Identity Matrix:** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

   i.e. $a_{ij} = \begin{cases} 0, \text{ when } i \neq j \\ 1, \text{ when } i = j \end{cases}$
8. **Upper Triangular Matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall\ i > j$.
9. **Lower Triangular Matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\ i < j$.
10. **Submatrix:** A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.
11. **Equal Matrices:** Two matrices A and B are said to be equal, if both having same order and corresponding elements of the matrices are equal.
12. **Principal Diagonal of a Matrix:** In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

    e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, the principal diagonal of A is 1, 6, 2.
13. **Singular Matrix:** A square matrix A is said to be singular matrix, if determinant of A denoted by det (A) or $|A|$ is zero, i.e. $|A| = 0$, otherwise it is a non-singular matrix.

Equality of Matrices

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

1. Both have the same order.
2. $a_{ij} = b_{ij}$ for each pair of i & j.

Algebra of Matrices

Addition: $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

1. **Addition of matrices is commutative:** $A + B = B + A$.
2. **Matrix addition is associative:** $(A + B) + C = A + (B + C)$.

Multiplication of a Matrix By a Scalar

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Multiplication of Matrices (Row by Column)

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m\times n} = [a_{ij}]$ and $B_{n\times p} = [b_{ij}]$, then order of AB is $m \times p$ and

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Characteristic Equation

Let A be a square matrix. Then the polynomial $| A - xI |$ is called as characteristic polynomial of A & the equation $| A - xI | = 0$ is called characteristic equation of A.

Properties of Matrix Multiplication

1. If A and B are two matrices such that
   (i) $AB = BA$ then A and B are said to commute
   (ii) $AB = -BA$ then A and B are said to anticommute
2. **Matrix Multiplication is Associative:** If A, B & C are conformable for the product AB & BC, then $(AB)C = A(BC)$.
3. **Distributivity:** $\left.\begin{array}{l} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{array}\right]$, provided A, B and C are conformable for respective products.

Positive Integral Powers of a Square Matrix

1. $A^m A^n = A^{m+n}$
2. $(A^m)^n = A^{mn} = (A^n)^m$
3. $I^m = I$ $m, n \in N$

Orthogonal Matrix

A square matrix is said to be orthogonal matrix if $AA^T = I$.

Key Note

- The determinant value of orthogonal matrix is either 1 or –1.
  Hence orthogonal matrix is always invertible.
- $AA^T = I = A^T A$. Hence $A^{-1} = A^T$.

Some Square Matrices

1. **Idempotent Matrix:** A square matrix is idempotent provided $A^2 = A$. For idempotent matrix note the following:
   (a) $A^n = A \ \forall \ n \geq 2, n \in N$.
   (b) determinant value of idempotent matrix is either 0 or 1.
   (c) If idempotent matrix is invertible then its inverse will be identity matrix i.e. I.
2. **Periodic Matrix:** A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.
   Note that period of an idempotent matrix is 1.
3. **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$.
   Note that a nilpotent matrix will not be invertible.
4. **Involutary Matrix:** If $A^2 = I$, the matrix is said to be an involutary matrix.
   Note that $A = A^{-1}$ for an involutary matrix.
5. If A and B are square matrices of same order and $AB = BA$ then
   $(A+B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1}B + {}^nC_2 A^{n-2}B^2 + \ldots + {}^nC_n B^n$.

Transpose of a Matrix (Changing Rows & Columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ij}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$.

Properties of Transpose

If A^T & B^T denote the transpose of A and B

1. $(A + B)^T = A^T + B^T$; note that A & B have the same order.
2. $(A\,B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB
3. $(A^T)^T = A$
4. $(kA)^T = kA^T$, where is a scalar.

General: $(A_1 \cdot A_2, \ldots A_n)^T = A^T{}_n \cdot \ldots A^T{}_2 \cdot A^T{}_1$ (reversal law for transpose)

Symmetric & Skew Symmetric Matrix

1. **Symmetric matrix:** For symmetric matrix $A = A^T$.
   Note: Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.
2. **Skew symmetric matrix:** Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji} \ \forall \ i$ & j. Hence if A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall \ i$.
   Thus the diagonal elements of a skew square matrix are all zero, but not the converse.
   For a skew symmetric matrix $A = -A^T$.
3. Properties of symmetric & skew symmetric matrix:
   (a) Let A be any square matrix then, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
   (b) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.
   (c) If A & B are symmetric matrices then,
   (i) $AB + BA$ is a symmetric matrix.
   (ii) $AB - BA$ is a skew symmetric matrix.
4. Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and $$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Adjoint of a Square Matrix

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then $(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$.

Key Note

If A be a square matrix of order n, then

1. $A(\text{adj } A) = |A|\, I_n = (\text{adj } A) \,.\, A$

2. $|\text{adj } A| = |A|^{n-1}$

3. $\text{adj}(\text{adj } A) = |A|^{n-2} A$

4. $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

5. $\text{adj } (AB) = (\text{adj } B)\,(\text{adj } A)$

6. $\text{adj } (KA) = K^{n-1}\,(\text{adj } A)$, where K is a scalar

Inverse of a Matrix (Reciprocal Matrix)

A square matrix A (non singular) said to be invertible, if there exists a matrix B such that, $AB = I = BA$.

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$$A^{-1} = B \quad \Leftrightarrow \quad AB = I = BA$$

We have, $A \cdot (\text{adj } A) = |A|\, I_n$

$$A^{-1} \cdot A(\text{adj } A) = A^{-1}\, I_n\, |A|$$

$$I_n\,(\text{adj } A) = A^{-1}\, |A|\, I_n$$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Theorem: If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Key Note

- If A be an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- If A is invertible,
  (a) $(A^{-1})^{-1} = A$
  (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}\,;\, k \in N$

System of Equation and Criteria for Consistency

Gauss - Jordan Method

Example:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}\, B = \frac{\text{Adj } A}{|A|} \cdot B$$

Key Note

- If $|A| \neq 0$, system is consistent having unique solution.
- If $|A| \neq 0$ and $(\text{adj } A) \cdot B \neq O$ (Null matrix), system is consistent having unique non-trivial solution.
- If $|A| \neq 0$ and $(\text{adj } A) \cdot B = O$ (Null matrix), system is consistent having trivial solution.
- If $|A| = 0$, then

Matrix Method Fails

If $(\text{adj } A) \cdot B = O$ (null matrix) → Consistent (infinite solutions)

If $(\text{adj } A) \cdot B \neq O$ → Inconsistent (no solutions)

CHAPTER

4 Determinants

Definition

1. The determinant consisting two rows and two columns is $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, its value is given by:

   $D = a_1b_2 - a_2b_2$
2. A determinant which consists of three rows and three columns is called a third-order-determinant.

   Let D = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then its value is

   $D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the minor M_{ij} of the element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column,

i.e. M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Properties of Minors and Cofactors

1. The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

   i.e., if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

   $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.
2. The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ,

   i.e., if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

   $|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.
3. In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ and $|(\text{adj } A)| = \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Properties of Determinants

1. The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.

   e.g., $|A'| = |A|$
2. If $A = [a_{ij}]_{n\times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then

   $\det(B) = -\det(A)$
3. If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.
4. $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.
5. $|kA| = k^n |A|$, where A is a matrix of order $n \times n$.
6. If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinant, e.g.,

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

7. If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g.,

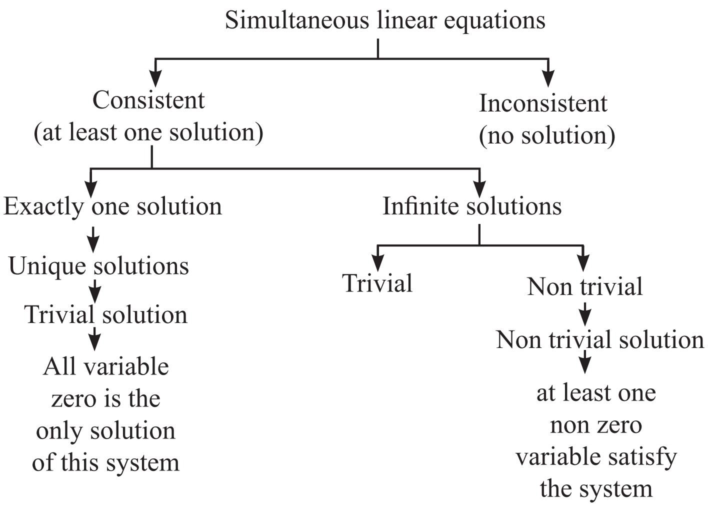
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

8. If each element of a row (or column) of a determinant is zero, then its value is zero.
9. If any two rows (or columns) of a determinant are identical, then its value is zero.
10. If r rows (or r columns) become identical, when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

Important Results on Determinants

1. $|AB| = |A||B|$, where A and B are square matrices of the same order.
2. $|A^n| = |A|^n$.
3. If A, B and C are square matrices of the same order such that i^{th} columns (or rows) of A is the sum of i^{th} columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$.
4. $|I_n| = 1$, where I_n is identity matrix of order n.
5. $|O_n| = 0$, where O_n is a zero matrix of order n.
6. If $\Delta(x)$ has a third order determinant having polynomials as its elements.
   (*a*) If $\Delta(a)$ has two rows (or columns) proportional, then $(x - a)$ is a factor of $\Delta(x)$.
   (*b*) If $\Delta(a)$ has three rows (or columns) proportional, then $(x - a)^2$ is a factor of $\Delta(x)$.
7. A square matrix A is non-singular, if $|A| \neq 0$ and singular, if $|A| = 0$.
8. Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.
9. In general, $|B + C| \neq |B| + |C|$.
10. Determinant of a diagonal matrix = Product of its diagonal elements.
11. If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.
12. Determinant of a orthogonal matrix = 1 or – 1.
13. Determinant of a hermitian matrix is purely real.
14. If A and B are non-zero matrices and $AB = O$, then it implies $|A| = 0$ or $|B| = 0$.

System of Equation



Cramer's Rule: [Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ...(*i*)

$a_2x + b_2y + c_2z = d_2$ ...(*ii*)

$a_3x + b_3y + c_3z = d_3$ ...(*iii*)

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, Z = \frac{D_3}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; D_2 \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\& D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note:

1. If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.
2. If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only
3. If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solution.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

1. Area of $\Delta ABC = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$

   $= \left| \frac{1}{2} [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)] \right|$

2. If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

3. Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

CHAPTER

5 Limits of Functions

Limit

Limit of a function $f(x)$ is said to exist as $x \to a$ when, $\underset{\substack{x \to a^- \\ \text{(Left hand limit)}}}{\lim} f(a-h) = \underset{\substack{x \to a^+ \\ \text{(Right hand limit)}}}{\lim} f(a+h) = M$ some finite value M.

Indeterminate Forms

$$\frac{0}{0} \quad , \quad \frac{\infty}{\infty} \quad , \quad (\infty)-(\infty)$$

$$\infty \times 0 \quad , \quad (1)^{\infty} \quad , \quad (0)^0 \quad , \quad (\infty)^0$$

Standard Limits

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{\sin^{-1}x}{x}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{ln(1+x)}{x} = 1,$$

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e, \lim_{x\to 0}\frac{a^x-1}{x} = \log_e a, a>0,$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}.$$

Fundamental Theorems on Limits

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exists finitely then:

(*a*) Sum rule: $\lim_{x\to a}[f(x)+g(x)] = l+m$

(*b*) Difference rule: $\lim_{x\to a}[f(x)-g(x)] = l-m$

(*c*) Product rule: $\lim_{x\to a}[f(x)\cdot g(x)] = l.m$

(*d*) Quotient rule: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(*e*) Power rule: If m and n are integers, then

$\lim_{x\to a}[f(x)]^{m/n} = l^{m/n}$, provided $l^{m/n}$ is a real number.

(*f*) $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(m)$; provided $f(x)$ is continuous at $x = m$.

Limits Using Expansion

(*i*) $a^x = 1 + \frac{xlna}{1!} + \frac{x^2 ln^2 a}{2!} + \frac{x^3 ln^3 a}{3!} + \ldots, a>0$

(*ii*) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$, for $-1 < x \le 1$

(*iii*) $ln\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$, for $-1 < x \le 1$

(*iv*) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(*v*) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(*vi*) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(*vii*) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2\cdot 3^2}{5!}x^5 + \frac{1^2\cdot 3^2\cdot 5^2}{7!}x^7 + \ldots$

(*viii*) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

(*xi*) For $|x|<1, n \in R, (1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}$

$x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\infty$

(*xii*) $(1+x)^{1/x} = e^{\frac{1}{x}ln(1+x)} = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \frac{21}{48}x^3 + \ldots + \infty\right]$

Limits of form 1^∞, $0°$, $\infty°$.

Also for $(1)^\infty$ type of problems we can use following rules.

(*a*) $\lim_{x\to 0}(1+x)^{1/x} = e,$

(*b*) $\lim_{x\to a}[f(x)]^{g(x)}$, where $f(x)\to 1; g(x)\to\infty$ as $x \to a$ then

$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a}\{f(x)-1\}g(x)}$

Sandwich Theorem or Squeeze Play Theorem

If $f(x) \le g(x) \le h(x)\ \forall\ x$ and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$

CHAPTER

6 Continuity and Differentiability, Methods of Differentiation

Properties of Continuous Functions

Here we present two extremely useful properties of continuous functions;

Let $y = f(x)$ be a continuous function $\forall\, x \in [a, b]$, then following results hold true.

(*i*) f is bounded between a and b. This simply means that we can find real numbers m_1 and m_2 such $m_1 \le f(x) \le m_2\ \forall\, x \in [a, b]$.

(*ii*) Every value between $f(a)$ and $f(b)$ will be assumed by the function atleast once. This property is called intermediate value theorem of continuous function.

In particular if $f(a) \cdot f(b) < 0$, then $f(x)$ will become zero atleast once in (a, b). It also means that if $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ will have atleast one real root in (a, b).

Types of Discontinuities

Type-1 : (Removable type of discontinuities)

(***a***) **Missing point discontinuity:** Where $\lim_{x\to a} f(x)$ exists finitely but $f(a)$ is not defined.

(***b***) **Isolated point discontinuity :** Where $\lim_{x\to a} f(x)$ exists & $f(a)$ also exists but; $\lim_{x\to a} f(x) \ne f(a)$.

Type-2 : (Non-Removable type of discontinuities)

(***a***) **Finite type discontinuity :** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(***b***) **Infinite type discontinuity :** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(***c***) **Oscillatory type discontinuity :** Limits oscillate between two finite quantities.

Derivability of Function at a Point

If $f'(a^+) = f'(a^-)$ = finite quantity, then $f(x)$ is said to be **derivable or differentiable at $x = a$.** In such case $f'(a^+) = f'(a^-) = f'(a)$ and it is called derivative or differential coefficient of $f(x)$ at $x = a$.

Note:

(*i*) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(*ii*) If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).\, g(x)$ will also be derivable at $x = a$ and if $g(a) \ne 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

In short, for a function 'f':

Differentiable $\Rightarrow$ Continuous;

Not Continuous $\Rightarrow$ Not Differentiable

Continuous $\Rightarrow$ May or may not be Differentiable

Derivability Over an Interval

(*a*) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each and every point of the open interval (a, b).

(*b*) $f(x)$ is said to be derivable over the closed interval $[a, b]$ if:

(*i*) $f(x)$ is derivable in (a, b) and

(*ii*) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Note:

(***i***) If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(***ii***) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(***iii***) If $f(x)$ & $g(x)$ both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

(***iv***) If $f(x)$ is derivable at $x = a \not\Rightarrow f'(x)$ is continuous at $x = a$.

Differentiation of Some Elementary Functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(a^x) = a^x \ln a$

3. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

10. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

Basic Theorems

1. $\frac{d}{dx}(f \pm g)(x) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$

3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

5. $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))g'(x)$

Derivative of inverse Trigonometric Functions

$$\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{d \cos^{-1} x}{dx} = -\frac{1}{\sqrt{1-x^2}}, \text{for } -1 < x < 1.$$

$$\frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}, \frac{d \cot^{-1} x}{dx} = -\frac{1}{1+x^2} \quad (x \in R)$$

$$\frac{d \sec^{-1} x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \frac{d\, \text{cosec}^{-1} x}{dx}$$

$$= -\frac{1}{|x|\sqrt{x^2-1}}, \text{for } x \in (-\infty, -1) \cup (1, \infty)$$

Differentiation Using Substitution

Following substitutions are normally used to simplify these expression.

1. $\sqrt{x^2 + a^2}$ by substituting $x = a \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

2. $\sqrt{a^2 - x^2}$ by substituting $x = a \sin \theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

3. $\sqrt{x^2 - a^2}$ by substituting $x = a \sec \theta$, where $\theta \in [0, \pi]$, $\theta \ne \frac{\pi}{2}$

4. $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a \cos \theta$, where $\theta \in [0, \pi]$.

Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Derivative of one Function with Respect to Another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

❖ If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

CHAPTER

7 Applications of Derivatives

When Limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change of y w.r.t. time at an instant.

i.e., $\lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t} = \dfrac{dy}{dt}$.

$\left(\dfrac{dy}{dx}\right)_P = \tan\theta$ = slope of tangent at P.

Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\dfrac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

Note:

1. If tangent is parallel to x-axis, $\theta = 0° \Rightarrow \tan\theta = 0$

   $\therefore \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 0$

2. If tangent is perpendicular to x-axis (or parallel to y-axis) then $\theta = 90° \Rightarrow \tan\theta \to \infty$ or $\cot\theta = 0$

   $\therefore \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

Equation of tangent and normal in parametric form

Let the equation of the curve be expressed in the parameteric form $x = g(t)$ and $y = \phi(t)$ where t is the parameter.

The equation of the tangent at a point $P(t)$,

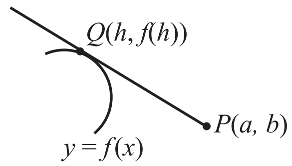
$$y - \phi(t) = \frac{\phi'(t)}{g'(t)}[x - g(t)] \quad \text{and}$$

the equation of normal is $y - \phi(t) = \dfrac{-g'(t)}{\phi'(t)}[x - g(t)]$

Tangent from an External Point

Given a point $P(a, b)$ which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve passing through (a, b) can be found by solving for the point of contact Q.

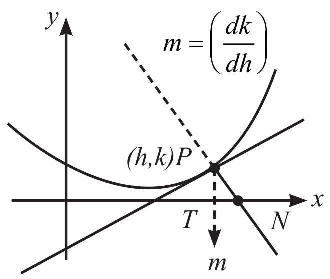
$$f'(h) = \frac{f(h) - b}{h - a}$$



And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

Length of Tangent, Normal, Subtangent, Subnormal at P(h,k)

1. $PT = |k| \sqrt{1 + \dfrac{1}{m^2}}$ = Length of Tangent
2. $PN = |k| \sqrt{1 + m^2}$ = Length of Normal
3. $TM = \left|\dfrac{k}{m}\right|$ = Length of subtangent
4. $MN = |km|$ = Length of subnormal.



Angle Between the Curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

If $\theta = \pi/2$, then the two curves are said to cut each other orthogonally and the condition for this to happen is:

$$m_1 \times m_2 = -1 \Rightarrow f'(x_0) \times g'(x_0) = -1$$

Shortest Distance between two Curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

Errors and Approximations

1. **Errors:** Let $y = f(x)$

   From definition of derivative, $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dx}$

   $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ approximately or $\Delta y = \left(\frac{dy}{dx}\right) . \Delta x$ approximately

   Definition:

   (*i*) Δx is known as **absolute error** in x.

   (*ii*) $\frac{\Delta x}{x}$ is known as **relative error** in x.

   (*iii*) $\frac{\Delta x}{x} \times 100\%$ is known as **percentage error** in x.

2. **Approximations:** From definition of derivative,

   As Derivative of $f(x)$ at $(x = a) = f'(a)$

   or $f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}$

   or $\frac{f(a+\delta x) - f(a)}{\delta x} \to f'(a)$ (approximately)

   $f(a + \Delta x) - f(a) \to \Delta x f'(a)$ (approximately).

Properties of Monotonic Functions

1. If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.
2. If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.
3. If $f(x)$ and $g(x)$ both are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing (in either case) function on $[a, b]$.
4. If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing (in either case) on $[a, b]$.
5. If $f(x)$ is increasing function then $\frac{1}{f(x)}$ is decreasing function for $f(x) \neq 0$.
6. If a function is invertible it has to be either increasing or decreasing.

Rolle's Theorem

If *a* function *f* defined on $[a, b]$ is

1. Continuous on $[a, b]$
2. derivable on (a, b) and
3. $f(a) = f(b)$.

Then there exists atleast one c $(a < c < b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem (LMVT)

If *a* function *f* defined on $[a, b]$ is

1. continuous on $[a, b]$ is
2. derivable on (a, b)
3. $f(a) = f(b)$,

then there exists at least one real numbers between a and b $(a < c < b)$ such

that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Special Points

1. **Critical points:** The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.
2. **Stationary points:** The stationary points are the points of domain where $f'(x) = 0$.

Note: Every stationary point is a critical point but vice-versa is not true.

Significance of the Sign of 2nd order Derivative and Point of Inflection

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b). Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b).

Useful Formulae of Mensuration to Remember

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3.
4. Surface area of cube = $6a^2$.
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height).
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base).
    (Note that lateral surfaces of a prism are all rectangle.)
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
    (Note that slant surfaces of a pyramid are triangles).

CHAPTER

8 Indefinite Integration

1. If f & g are functions of x such that $g'(x)=f(x)$ then,

$\int f(x)dx = g(x)+c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the constant of integration.

2. Standard Formula:

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c, n\neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a} ln\,|ax+b|+c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{ln\,a}+c; a>0$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a} ln \sec|(ax+b)|+c$

(viii) $\int \cot\,(ax+b)dx = \frac{1}{a} ln\,|\sin(ax+b)|+c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+c$

(x) $\int \text{cosec}^2\,(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

(xi) $\int \sec x\,dx = ln\,|(\sec x+\tan x)|+c$

Or $ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+c$

(xii) $\int \text{cosec}\,x\,dx = ln\,|(\text{cosec}\,x-\cot x)|+c$

Or $ln\left|\tan\frac{x}{2}\right|+c$

Or $-ln\,|(\text{cosec}\,x+\cot x)|+c$

(xiii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

(xiv) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

(xv) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xvi) $\int \frac{dx}{\sqrt{x^2+a^2}} = ln\left|x+\sqrt{x^2+a^2}\right|+c$

(xvii) $\int \frac{dx}{\sqrt{x^2-a^2}} = ln\left|x+\sqrt{x^2-a^2}\right|+c$

(xviii) $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}ln\left|\frac{a+x}{a-x}\right|+c$

(xix) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}ln\left|\frac{x-a}{x+a}\right|+c$

(xx) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxi) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|+c$

(xxii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c$

3. Integration by substitutions:

If we substitute $f(x)=t$, then $f'(x)\,dx = dt$

4. Integration by part:

$$\int\big(f(x)g(x)dx = f(x)\big)\int(g(x))dx - \int\left(\frac{d}{dx}(f(x))\int(g(x))dx\right)dx$$

5. Integration of type:

$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,bx$

Make the substitution $x+\frac{b}{2a}=t$

6. Integration of type:

$\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,$

$\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Make the substitution $x + \frac{b}{2a} = t$, then split the integral as sum of two integrals one containing the linear term and the other containing constant term.

7. Integration of trigonometric functions:

(*i*) $\int \frac{dx}{a+b\sin^2 x}$ Or, $\int \frac{dx}{a+b\cos^2 x}$ Or,

$\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$, put $\tan x = t$

(*ii*) $\int \frac{dx}{a+b\sin x}$ Or, $\int \frac{dx}{a+b\cos x}$ Or

$\int \frac{dx}{a+b\sin x + c\cos x}$, put $\tan\frac{x}{2} = t$.

(*iii*) $\int \frac{a.\cos x + b.\sin x + c}{l.\cos x + m.\sin x + n} dx$.Express

$N^r \equiv A(D^r) + B\frac{d}{dx}(D^r) + K$ & proceed.

8. Integration of type:

$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1} dx$ where K is any constant.

Divide N^r & D^r by x^2 & put $x \mp \frac{1}{x} = t$

9. Integration of type:

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ Or $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$;

put $px + q = t^2$.

10. Integration of type:

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b = \frac{1}{t}$;

$\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ put $x = \frac{1}{t}$

Some Standard Substitution

1. $\int (f(x))^n f'(x)dx$ Or $\int \frac{f'(x)}{[f(x)]^n} dx$ put $f(x) = t$ & proceed.

2. $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

3. $\int \frac{(px+q)dx}{ax^2+bx+c}, \int \frac{(px+q)}{\sqrt{ax^2+bx+c}} dx$

Express $px + q = A$ (differential coefficient of denominator) $+ B$.

4. $\int e^x[f(x)+f'(x)]dx = e^x.f(x)+c$

5. $\int [f(x)+xf'(x)]dx = xf(x)+c$

6. $\int \frac{dx}{x(x^n+1)}, n \in N,$ take x^n common & put $1 + x^{-n} = t$.

7. $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}, n \in N,$ take x^n common & put $1 + x^{-n} = t^n$.

8. $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t$.

9. $\int \sqrt{\frac{x-\alpha}{\beta-x}} dx$ Or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}} dx$ Or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

CHAPTER

9 Definite Integration

The Fundamental Theorem of Calculus Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Note: If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

✦ $\int_a^b f(x)dx$ = algebraic area under the curve $f(x)$ from a to b

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_{-a}^a f(x)dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)dx, & f(-x)=f(x) \\ 0, & f(-x)=-f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x))dx$

6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

7. $\int_0^{2a} f(x)dx = \int_0^a (f(x)+f(2a-x))\,dx$

$$= \begin{cases} 2\int_0^a f(x)dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

8. If $f(x)$ is a periodic function with period T, then

$$\int_0^{nT} f(x)dx = n\int_0^T f(x)\,dx, n \in Z,$$

$$\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in Z, a \in R$$

$$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m, n \in Z,$$

$$\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in Z, a \in R$$

$$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx, n \in Z, a, b \in R$$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \Psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx$$

Leibnitz Theorem

If $F(x) = \int_{g(x)}^{h(x)} f(t)dt,$

then $\frac{dF(x)}{dx} = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

Walli's Formula

1. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)}K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

2. $\int_0^{\pi/2} \sin^n x . \cos^m x \, dx$

$$= \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2} K$$

$$\text{where } K = \begin{cases} \frac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

Definite Integral as Limit of a Sum

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$$

$$\Rightarrow \quad \lim_{h \to \infty} h \sum_{r=0}^{n-1} f(a+rh) = \int_0^1 f(x)dx \text{ where } b - a = nh$$

If $a = 0$ and $b = 1$ then, $\lim_{n \to \infty} h \sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

OR $\lim_{n \to \infty} \left(\frac{1}{n}\right) \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$

Estimation of Definite Integral

1. If $f(x)$ is continuous in $[a, b]$ and it's range in this interval is $[m, M]$, them $m\,(b - a) \le \int_a^b f(x)dx \le M(b-a)$

2. If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

3. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx.$

4. If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)dx \ge 0.$

5. $f(x)$ and $g(x)$ are two continuous function on $[a, b]$ then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}$$

Some Standard Results

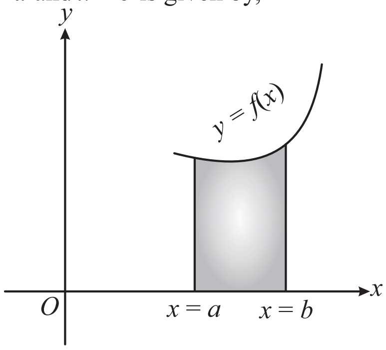
1. $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2 = \int_0^{\pi/2} \log \cos x \, dx$

2. $\int_a^b \frac{|x|}{x} dx = |b| - |a|.$

CHAPTER

10 Application of Integrals

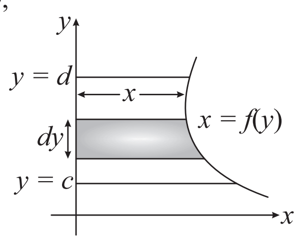
1. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by,



$$A = \int_a^b f(x)dx = \int_a^b y\,dx$$

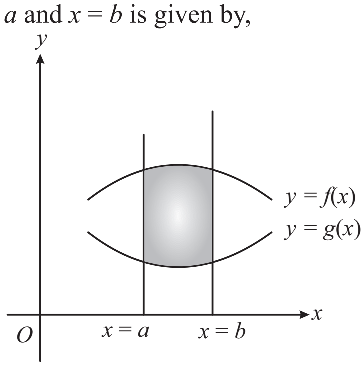
2. If the area is below the x-axis, then A is negative. The convention is to consider the magnitude only *i.e.* $A = \left|\int_a^b y\,dx\right|$ in this case.

3. The area bounded by the curve $x = f(y)$, y-axis and abscissa $y = c$, $y = d$ is given by,



$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)dy$$

4. Area between the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$ is given by,



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]\,dx$$

5. Average value of a function $y = f(x)$ *w.r.t.* x over an interval $a \le x \le b$ is defined as:

$$y_{av} = \frac{1}{b-a}\int_a^b f(x)dx.$$

6. **Curve Tracing:**

The following outline procedure is to be applied in Sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(***a***) **Symmetry:** The symmetry of the curve is judged as follows:

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x and y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x and y both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(***b***) Find dy/dx and equate it to zero to find the points on the curve where you have horizontal tangents.

(***c***) Find the points where the curve crosses the x-axis and also the y-axis.

(***d***) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. **Useful Results:**

(***a***) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is $\pi\, ab$.

(***b***) Area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $16ab/3$.

(***c***) Area included between the parabola $y^2 = 4\,ax$ and the line $y = mx$ is $\frac{8}{3}\frac{a^2}{m^3}$

CHAPTER

11 Differential Equations

Order of a Differential Equation

The order of highest order derivative appearing in a differential equation is called the order of the differential equation.

Degree of a Differential Equation

The degree of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives.

Variable Separable Differentiable Equations

A differential equation of the form $f(x) + g(y)\dfrac{dy}{dx} = 0$

Equations Reducible to Variable Separable form

$\dfrac{dy}{dx} = f\,(ax + by + c)$ can be reduced to varible separable form by substitution

$ax + by + c = t$. The reduced variable separable form is:

$$\frac{dt}{bf(t) + a} = dx.$$

Homogeneous Differential Equation

$\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous function of same degree in x and y.

Substitute $y = vx$ and so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equations Reducible to the Homogeneous form

Consider a differential equation of the form:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Put $x = X + h$

$y = Y + k$

Such that, $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$

Linear Equation

An equation of the form $\dfrac{dy}{dx} + Py = Q$

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \quad \frac{dy}{dx}e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

$$\Rightarrow \quad y.e^{\int Pdx} = \int Q.e^{\int Pdx}dx + C$$

Bernoulli's Equation

An equation of the form $\dfrac{dy}{dx} + Py = Qy^n$,

Putting $\quad y^{-n+1} = v$

$$\Rightarrow \quad \frac{dv}{dx} + (1 - n)\,P\cdot\ y = (1 - n)Q.$$

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d\left(\ell nxy\right)$

(v) $\dfrac{dx + dy}{x + y} = d(\ell n(x + y))$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(*ix*) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(*x*) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(*xi*) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2 y^2}$

(*xii*) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

Orthogonal Trajectory

Any curve which cuts every member of a given family of curves at right angle is called an orthogonal trajectory of the family. For example, each straight line $y = mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for Finding the Orthogonal Trajectory

(*i*) Let $f(x, y, c) = 0$ be the equation, where c is an arbitrary parameter.

(*ii*) Differentiate the given equation w.r.t. x and then eliminate c.

(*iii*) Replace $\dfrac{dy}{dx} by -\dfrac{dx}{dy}$ in the equation obtained in (*ii*).

(*iv*) Solve the differential equation in (*iii*).

CHAPTER

12 Vector Algebra

Important Definitions

- **Representation of Vectors:** A vector $\vec{a}$ is represented by the directed line segment $\overrightarrow{AB}$. The magnitude of the vector $\vec{a}$ is equal to $|\overrightarrow{AB}|$, and the direction of the vector $\vec{a}$ is along the line from A to B.
- **Scalar Quantity:** A quantity that has only magnitude and is not related to any direction is called a scalar quantity.
- **Vector Quantity:** A quantity that has magnitude and also a direction in space is called a vector quantity.
- **Null Vector or Zero Vector:** If the initial and terminal points of a vector coincide, then it is called a zero vector. It is denoted by $\vec{0}$ or O. Its magnitude is zero and direction indeterminate.
- **Unit Vector:** A vector whose magnitude is of unit length along my vector $\vec{a}$ is called a unit vector in the direction of $\vec{a}$ and is denoted by $\hat{a}$
- **Equal Vector:** Two non-zero vectors are said to be equal vectors if their magnitude is equal and directions are the same.
- **Collinear Vector:** Two or more non-zero vectors are said to be collinear vectors if they are parallel to the same line.
- **Like and Unlike Vector:** Collinear vectors having the same direction are known as like vectors, while those having opposite directions are known as, unlike vectors.
- **Coplanar Vector:** Two or more non-zero vectors are said to be coplanar vectors if these are parallel to the same plane.
- **Localised Vector and Free Vector:** A vector drawn parallel to a given vector through a specified point as the initial point, is known as a localised vector. If the initial point of a vector is not specified, it is said to be a free vector.
- **Position Vector:** Let O be the origin and A be a point such that $\overrightarrow{OA} = \vec{a}$, then we say that the position vector of A is $\vec{a}$.

Negative of a Vector

- Let $\overrightarrow{AB}$ be a vector directed from A to B. then $-\overrightarrow{AB}$ is a vector which would be directed from B to A.

Coinitial Vectors

- Two vectors are said to be coinitial vectors if both the vectors have the same initial points.

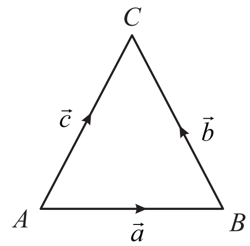
Co-terminal Vectors

- Two vectors are said to be Co-terminal vectors if both the vectors have the same terminating point.

Algebra of vectors

Addition of Vectors

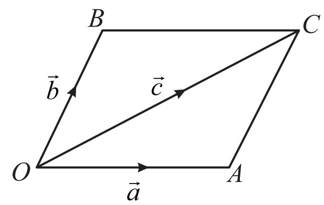
Triangle Law



Result: $\vec{a} + \vec{b} = \vec{c}$ or $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Converse of triangle law is also true.

Parallelogram Law



Result: $\vec{a} + \vec{b} = \vec{c}$ or $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

Properties of vector addition:

(*i*) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative)

(*ii*) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative)

(*iii*) $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$

(*iv*) $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$

(*v*) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

(*vi*) $|\vec{a} - \vec{b}| \ge ||\vec{a}| - |\vec{b}||$

Multiplication of Vector by Scalars

If $\vec{a}$ and $\vec{b}$ are vectors & m, n are scalars, then

(*i*) $m(\vec{a}) = (\vec{a})m = m\vec{a}$

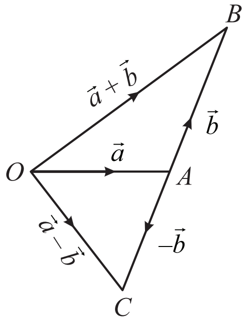
(*ii*) $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(*iii*) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

(*iv*) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Subtraction of Vectors

In the given diagram $\vec{a}$ and $\vec{b}$ are represented by $\overline{OA}$ and $\overline{AB}$. We extend the line AB in opposite direction upto C, where $AB = AC$. The line segment $\overline{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overline{OC}$ is $\vec{a}+(-\vec{b})$. i.e., denotes the vector $\vec{a}-\vec{b}$.



Note:

(*i*) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

(*ii*) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(*iii*) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtaction of vectors does not obey the associative law.

Important Properties and Formulae

- If $\vec{r}_1 = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ then $\vec{r}_1+\vec{r}_2=(x_1+x_2)\hat{i}+(y_1+y_2)\hat{j}+(z_1+z_2)\hat{k}$ and $\vec{r}_1=\vec{r}_2$ $\Leftrightarrow$ $x_1=x_2,\ y_1=y_2,\ z_1=z_2$.
- $\vec{a}$ and $\vec{b}$ are parallel or collinear if $\vec{a}=m\vec{b}$ and only if for some non-zero scalar m.
- $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$ or $\vec{a}=|\vec{a}|\,\hat{a}$
- $\vec{r},\vec{a},\vec{b}$ are coplanar if and only if $\vec{r}=x\vec{a}+y\vec{b}$ for some scalars x and y.
- If the position vectors of the points A and B be $\vec{a}$ and $\vec{b}$ then, the position vectors of the points dividing the line AB in the ratio $m : n$ internally and externally are $\dfrac{m\vec{b}+n\vec{a}}{m+n}$ and $\dfrac{m\vec{b}-n\vec{a}}{m-n}$, respectively.
- If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then $|\vec{r}|=\sqrt{x^2+y^2+z^2}$
- Given vectors $x_1\vec{a}+y_1\vec{b}+z_1\vec{c},\ x_2\vec{a}+y_2\vec{b}+z_2\vec{c},\ x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$, where $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors, will be coplanar if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$
- $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$
- $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$

Scalar Product or Dot Product

- $\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta$, where $0\le\theta\le\pi$
- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$
- If $\vec{a}$ and $\vec{b}$ are the non-zero vectors, then $\vec{a}\cdot\vec{b}=0\Leftrightarrow\vec{a}\perp\vec{b}$
- $\cos\theta=\left|\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right|$ where θ is the acute angle made by $\vec{a}$ with $\vec{b}$
- Projection of $\vec{b}$ along $\vec{a}=\dfrac{\vec{b}.\vec{a}}{|\vec{a}|}$
- Component of a vector $\vec{r}$ in the direction of $\vec{a}$ and perpendicular to $\vec{a}$ are $\left(\dfrac{\vec{r}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{r}-\left\{\dfrac{(\vec{r}.\vec{a})}{|\vec{a}|^2}\right\}\vec{a}$ respectively.
- $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$ and $\hat{i}.\hat{j}=\hat{j}.\hat{i}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=\hat{i}.\hat{k}=0$

Vector Product

- The product of vectors $\vec{a}$ and $\vec{b}$ and is denoted by $\vec{a}\times\vec{b}=(|\vec{a}||\vec{b}|\sin\theta)\hat{n}$
- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
- If $\vec{a}=\vec{b}$ or if $\vec{a}$ is parallel to $\vec{b}$, then $\sin\theta=0$ and so $\vec{a}\times\vec{b}=0$
- Distributive laws: $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ and $(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$
- If $\vec{a}=a_1\hat{j}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

  (*i*) $\vec{a}\times\vec{b}=(a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$

  (*ii*) $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
- If two vectors $\vec{a}$ and $\vec{b}$ are parallel, then θ = 0 or π i.e. sin θ = 0 in both cases.
- Two vectors $\vec{a}$ and $\vec{b}$ are parallel if their corresponding components are proportional.
- Area of the triangle $ABC=\dfrac{1}{2}|\overline{AB}\times\overline{AC}|$

  $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0},\ \hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$
- Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}||\vec{b}|}$

Scalar Triple Product

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ *and* $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $(\vec{a} \times \vec{b})\cdot\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
- $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{b}\,\vec{a}]$ but $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$ etc.
- If any two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.
- The position of dot and cross in a scalar triple product can be interchanged. Hence, $(\vec{a} \times \vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b} \times \vec{c})$
- The value of a scalar triple product is zero if two of its vectors are parallel.
- $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.
- Volume of the parallelepiped whose coterminous edges are formed by $\vec{a}, \vec{b}, \vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]$.
- Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}\left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$.
- Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2}\left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$.
- In particular $\hat{i}.(\hat{j} \times \hat{k}) = 1$

  $[\hat{i}\,\hat{j}\,\hat{k}] = 1$
- $[K\,\vec{a}\,\vec{b}\,\vec{c}] = K[\vec{a}\,\vec{b}\,\vec{c}]$
- $[(\vec{a} + \vec{b})\,\vec{c}\,\vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}]$
- $[\vec{a} - \vec{b}\;\; \vec{b} - \vec{c}\;\; \vec{c} - \vec{a}] = 0$ and $[\vec{a} + \vec{b}\;\; \vec{b} + \vec{c}\;\; \vec{c} + \vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$
- $[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = [\vec{a} \times \vec{b}\;\; \vec{b} \times \vec{c}\;\; \vec{c} \times \vec{a}]$

Vector Triple Product

- If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are known as vector triple product.
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$
- $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of vectors $\vec{b}$ and $\vec{c}$.
- The vector triple product is not commutative i.e., $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$
- **Lagrange's identity:** $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

  $= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$
- $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

  $= [\vec{c}\,\vec{d}\,\vec{a}]\vec{b} - [\vec{c}\,\vec{d}\,\vec{b}]\vec{a}$

Distance between Lines

(*i*) If two parallel lines are given by

$\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, then distance ($d$) between them is given by

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

Shortest Distance $= \left|\frac{\overrightarrow{AB}.(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right| = \left|\frac{(\vec{b} - \vec{a}).(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

The two lines directed along $\vec{p}$ and $\vec{q}$ will intersect only if shortest distance = 0.

Reciprocal System of Vectors

- If $\vec{a}, \vec{b}, \vec{c}$ be any three non-coplanar vectors so that $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ then the three vectors $\vec{a}', \vec{b}', \vec{c}'$ defined by the equations $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a}, \vec{b}, \vec{c}$.
- **Properties of Reciprocal system of vectors:**

  (*i*) $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

  (*ii*) $[\vec{a}\,\vec{b}\,\vec{c}][\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$

  (*iii*) $\vec{i}' = \vec{i}, \vec{j}' = \vec{j}, \vec{k}' = \vec{k}$

  (*iv*) If $\{\vec{a}', \vec{b}', \vec{c}'\}$ is reciprocal system of $\{\vec{a}, \vec{b}, \vec{c}\}$ and $\vec{r}$ is any vector, then

  $\vec{r} = (\vec{r}.\vec{a})\vec{a}' + (\vec{r}.\vec{b})\vec{b}' + (\vec{r}.\vec{c})\vec{c}'$

  $\vec{r} = (\vec{r}.\vec{a}')\vec{a} + (\vec{r}.\vec{b}')\vec{b} + (\vec{r}.\vec{c}')\vec{c}$

CHAPTER

13 Three Dimensional Geometry

1. **Vector Representation of a Point:** Position vector of point $P(x, y, z)$ is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. **Distance Formula**

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, AB = |\overrightarrow{OB} - \overrightarrow{OA}|$$

3. **Distance of P from Coordinate Axes**

$PA = \sqrt{y^2 + z^2}, PB = \sqrt{z^2 + x^2}, PC = \sqrt{x^2 + y^2}$, A, B, C are on x, y, z axis respectively.

4. **Section Formula:** $x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}, z = \frac{mz_2 + nz_1}{m+n}$

Mid Point: $x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}, z = \frac{z_1 + z_2}{2}$

5. **Direction Cosines and Direction Ratios**

(*i*) **Direction cosines:** Let α, β, γ be angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (l, m, n). Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(*ii*) If l, m, n be the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.

(*iii*) **Direction ratios:** Let a, b, c be proportional to the direction cosines l, m, n then a, b, c are called the direction ratios.

(*iv*) If l, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(*v*) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $l = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

6. **Angle between Two Line Segments**

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

7. **Projection of a Line Segment on a Line:** If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines l, m, n is $|l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.

8. **Equation of a Plane:** General form : $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

(*i*) Normal form: $lx + my + nz = p$

(*ii*) Plane through the point (x_1, y_1, z_1):

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

(*iii*) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(*iv*) Vector form: $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

(*v*) Any plane parallel to the given plane

$ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

(*vi*) **Equation of a plane passing through a given point and parallel to the given vectors:** $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ (non parametric form)

9. **A Plane and a Point**

(*i*) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\frac{ax' + by' + cz' + d}{\sqrt{a^2 + b^2 + c^2}}$.

(*ii*) Length of the perpendicular from a point $(\vec{a})$ to plane $\vec{r} \cdot \vec{n} = d$ is given by $p = \frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$.

(*iii*) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$.

(*iv*) **To find image of a point w.r.t. a plane:** Let $P(x_1, y_1, z_1)$ is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image point then

$$\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -2\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}.$$

10. Angle between Two Planes: $\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$.

Planes are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$ (λ is a non zero scalar.)

11. Angle Bisectors

(*i*) The equations of the planes bisecting the angle between two given planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(*ii*) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. Family of Planes

(*i*) Any plane through the intersection of

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

(*ii*) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ *is* $\vec{r}\cdot(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Area of Triangle: From two vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

14. Volume of a Tetrahedron: Volume of a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$.

A Line

1. Equation of a Line

(*i*) A straight line is intersection of two planes.
It is represented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

(*ii*) **Symmetric form:** $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$.

(*iii*) **Vector equation:** $\vec{r} = \vec{a} + \lambda\vec{b}$.

(*iv*) Reduction of cartesion form of equation of a line to vector form and vice versa

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

2. Angle between a Plane and a Line

(*i*) If θ is the angle between line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, Then

$$\sin\theta = \left[\frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{l^2 + m^2 + n^2}}\right].$$

(*ii*) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b}) = \vec{r}.\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(*iii*) Condition for perpendicularity $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b}\times\vec{n} = 0$.

(*iv*) Condition for parallel $al + bm + cn = 0$, $\vec{b}.\vec{n} = 0$.

3. Condition for a Line to Lie in a Plane

(*i*) **Cartesian form:** $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $al + bm + cn = 0$.

(*ii*) **Vector form:** $\vec{r} = \vec{a} + \lambda\vec{b}$ would line in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$.

4. Skew Lines

(*i*) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. If $\begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \neq 0$. Then lines are skew.

(*ii*) **Vector Form:** For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to be skew $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) \neq 0$.

(*iii*) Shortest distance between line $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

CHAPTER

14 Linear Programming

- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called objective function) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). Variables are sometimes called decision variables and are non-negative.
- A few important linear programming problems are:
  (*i*) Diet problems
  (*ii*) Manufacturing problems
  (*iii*) Transportation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0$, $y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.
- Points within and on the boundary of the feasible region represent feasible solutions of the constraints.
- Any point outside the feasible region is an infeasible solution.
- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.
- The following Theorems are fundamental in solving linear programming problems:

Theorem 1: Let *R* be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When *Z* has an optimal value (maximum or minimum), where the variables *x* and *y* are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If *R* is bounded, then the objective function *Z* has both a maximum and a minimum value on *R* and each of these occurs at a corner point (vertex) of *R*.

- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of *R*.
- Corner point method for solving a linear programming problem. The method comprises of the following steps:
  (*i*) Find the feasible region of the linear programming problem and determine its corner points (vertices).
  (*ii*) Evaluate the objective function $Z = ax + by$ at each corner point. Let *M* and m respectively be the largest and smallest values at these points.
  (*iii*) If the feasible region is bounded, *M* and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded, then
  (*i*) *M* is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  (*ii*) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

CHAPTER

15 Probability

Mutually Exclusive Events

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any of the remaining events.

Thus, $E_1, E_2, ..., E_n$ are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$.

Independent Events

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, when a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

Complement of An Event

The complement of an event E, denoted by $\overline{E}$ or E' or E^c, is the set of all sample points of the space other than the sample points in E.

For example, when a die is thrown, sample space

$S = \{1, 2, 3, 4, 5, 6\}$.

If $E = \{1, 2, 3, 4\}$, then $\overline{E} = \{5, 6\}$.

Note that $E \cup \overline{E} = S$.

Mutually Exclusive and Exhaustive Events

A set of events $E_1, E_2, ..., E_n$ of a sample space S form a mutually exclusive and exhaustive system of events, if

(*i*) $E_i \cap E_j = \phi$ for $i \neq j$ and

(*ii*) $E_1 \cup E_2 \cup ... \cup E_n = S$

Notes:

(*i*) $O \leq P(E) \leq 1$, i.e. the probability of occurrence of an event is a number lying between 0 and 1.

(*ii*) $P(\phi) = 0$, i.e. probability of occurrence of an impossible event is 0.

(*iii*) $P(S) = 1$, i.e. probability of occurrence of a sure event is 1.

ODDs in Favour of an Event and ODDs Against An Event

If the number of ways in which an event can occur be m and the number of ways in which it does does not occur be n, then

(*i*) Odds in favour of the event = $\frac{m}{n}$ and

(*ii*) Odds against the event = $\frac{n}{m}$.

Some Important Results on Probability

1. $P(\overline{A}) = 1 - P(A)$.
2. If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0$.
   $\therefore P(A \cup B) = P(A) + P(B)$.
4. If A, B, C are any three events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.
5. If A, B, C are mutually exclusive events, then $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$, $A \cap B \cap C = \phi$ and hence $P(A \cap B) = 0$, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.
   $\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
6. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$.
7. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$.
8. $P(A) = P(A \cap B) + P(A \cap \overline{B})$.
9. $P(B) = P(B \cap A) + P(B \cap \overline{A})$.
10. If $A_1, A_2, ..., A_n$ are independent events, then $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2) ... P(A_n)$.
11. If $A_1, A_2, ..., A_n$ are mutually exclusive events, then
    $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$.
12. If $A_1, A_2, ..., A_n$ are exhaustive events, then $P(A_1 \cup A_2 \cup ... \cup A_n) = 1$.
13. If $A_1, A_2, ..., A_n$ are mutually exclusive and exhaustive events, then
    $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n) = 1$.
14. If $A_1, A_2, ..., A_n$ are n events, then
    (*i*) $P(A_1 \cup A_2 \cup ... \cup A_n) \leq P(A_1) + P(A_2) + ... + P(A_n)$.
    (*ii*) $P(A_1 \cap A_2 \cap ... \cap A_n) \geq 1 - P(\overline{A}_1) - P(\overline{A}_2) ... - P(\overline{A}_n)$.

Conditional Probability

$P\left(B/A\right)$ = Probability of occurrence of A, given that B has already happened.

$= \dfrac{P(A \cap B)}{P(B)} = \dfrac{n(A \cap B)}{n(B)}$

1. **Multiplication theorems on probability**
   (*i*) If A and B are two events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P\left(B/A\right)$, If $P(A) \neq 0$ or $P(A \cap B) = P(B) \cdot P\left(B/A\right)$, if $P(B) \neq 0$

   (*ii*) **Multiplication theorems for independent events:** If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P(B)$ i.e. the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A \cap B) = P(A) \cdot P(B/A)$. Since A and B are independent events, therefore

   $P(B/A) = P(B)$. Hence, $P(A \cap B) = P(A) \cdot P(B)$.

2. **Probability of at least one of the n independent events:** If $p_1, p_2, p_3, ...p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3, ... A_n$ respectively, then

   (*i*) Probability of happening none of them = $P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 ... \cap \overline{A}_n) = P(\overline{A}_1) \cdot P(\overline{A}_2) \cdot P(\overline{A}_3) ... P(\overline{A}_n)$
   $= (1 - p_1)(1 - p_2)(1 - p_3) ... (1 - p_n)$.

   (*ii*) Probability of happening at least one of them
   $= P(A_1 \cup A_2 \cup A_3 ... \cup A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3) ... P(\overline{A}_n)$
   $= 1 - (1 - p_1)(1 - p_2)(1 - p_3) ... (1 - p_n)$

Law of Total Probability

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + ... + P(E_n) \cdot P\left(\frac{A}{E_n}\right)$$

Baye's rule as $P(E_i/A) = \dfrac{P(E_i) \cdot P(A/E_i)}{\sum_{k=1}^{n} P(E_k)\, P(A/E_k)}$.

Binomial Distribution

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

Random Variable

The expectation (mean) of the random variable X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$ and the variance of X is defined as

$$\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2.$$